III. *Eddy-Current Losses in Cylindrical Conductors, with Special Applications to the Alternating Current Resistances of Short Coils.*

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INTRODUCTION.

IT is well-known that a considerable proportion of the effective resistance of inductive coils when used at radio frequencies is caused by the eddy-currents set up in the wires of the coils by the alternating magnetic field in which they are situated, and that in extreme cases the alternating current resistance may amount to more than one hundred times the direct current resistance. It is therefore important to have reliable formulæ for the eddy-current resistance of such coils in order to determine the conditions which will reduce the eddy-current losses to a minimum.

The simplest case, that of a long straight cylindrical wire under the action of its own current, has been treated by KELVIN,* RAYLEIGH,† HEAVISIDE,‡ and others. The general effect is known as the "skin effect," because the current tends to concentrate more and more upon the skin of the conductor as the frequency increases.

The case of two parallel wires forming a go-and-return circuit has been considered theoretically by NICHOLSON,§ and experimentally examined by KENNELLY.|| KENNELLY found that when the wires are close together, the added resistance due to the proximity of the wires may be of the same order as that due to the simple skin effect.

NICHOLSON's theoretical treatment includes the possibility that the dimensions of the system may be comparable with the wave-length of the disturbance. His formula is very complicated and difficult to apply numerically. A formula {formula (47)}

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* 'Math. and Phys. Papers,' vol. 3, 1889.

† 'Phil Mag.,' vol. 21, 1886.

‡ 'Electrical Papers,' vol. 2, p. 64.

§ 'Phil. Mag.,' vol. 18, p. 417, 1909.

|| 'Trans. A.I.E.E.,' vol. 35, part 2, p. 1953, 1915. CURTIS ('Bull. Bureau of Standards,' 1920) has recently published a formula for this case which gives agreement with KENNELLY's results.

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for this case is obtained in Section 8 of the present paper. This formula is shown (Section 9) to give results in close accordance with KENNELLY's observations.

In order to reduce the eddy-current losses solid wire is often replaced by stranded wire in which a bundle of thin separately insulated wires are interlaced symmetrically with each other, the notion being that the sum of the eddy losses in the individual wires shall be less than the eddy loss in the corresponding solid wire. LINDEMANN* verified this experimentally at certain frequencies, but also found that if a solid wire coil and stranded wire coil were compared at various frequencies, the stranded wire coil increased in resistance more slowly at the lower frequencies but less slowly at the high frequencies, until above a certain frequency the stranded wire coil had a greater effective resistance than the solid wire coil.

HOWE† has treated the problem of straight stranded wire conductors, assuming the eddy losses to increase as the square of the frequency, and from his formulæ has shown that at high frequencies it is difficult to make the resistance of the stranded wire less than that of solid wire of equal section.

In view of the extensive use of stranded wire in the construction of coils for high-frequency currents it is important that the limitations of stranded wire in reducing effective resistance should be known, so that the present investigation includes the consideration of such coils. From the formulæ obtained, conclusions are drawn in regard to the utility of stranding and in regard to what degree of stranding it is necessary to employ, before any improvement over solid wire coils may be expected.

In formulæ hitherto given for the effective resistance of coils, one or other of the following limitations occur:—

(1) The coil is very long.
(2) The frequency is limited to so low a value that the "square of frequency" law holds.
(3) The coil is wound with wire of square section.

The formulæ deduced in this paper differ from those already established in that—

(1) The dimensions of the winding sections of the coils are small compared with the coil radii.
2) There is no limitation imposed upon the frequency.
(3) The wire is taken to be circular.

In regard to (1) it is shown that coils of this type have better alternating-current time constants than long coils.

In regard to variation with frequency, the factor governing the upper limit to the application of the square law is the magnitude of f/R_0 where f is the frequency and R_0 is the direct current resistance per unit length of the wire used. If (in C.G.S. units)

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* 'Deut. Phys. Gesell.,' 1909, p. 382; 1910, p. 572. 'Jahrbuch der Drahtlosen Telegraphie,' 1911, p. 561.

† 'Roy. Soc. Proc.,' A, vol. 93, p. 468, 1917.

f/R_0 is less than 0·225 the eddy-current losses vary as $(f/R_0)^2$ to an accuracy of one per cent. At higher frequencies the variation is slower, the ultimate rate of variation being as $(f/R_0)^{\frac{1}{2}}$. A knowledge of these limiting rates of variation enables an immediate explanation to be given of LINDEMANN's results with stranded wire coils.

A solid wire in a given alternating field has eddy losses which are a function of $f/R_0 = \phi(f/R_0)$ say. If the solid wire is replaced by s strands of the same total metallic section, the loss per strand in the same field is $\phi(f/sR_0)$ and the total loss in the s strands is $s\phi(f/sR_0)$.

Thus, as regards the losses due to the general field of the remainder of the coil, we must replace $\phi(f/R_0)$ by $s\phi(f/sR_0)$ in passing from solid to stranded wire.

At low frequencies $\phi(f/R_0) = C(f/R_0)^2$ where C is a constant independent of the stranding, so that the respective losses are $C(f/R_0)^2$ and $Cs(f/sR_0)^2 = C(f/R_0)^2/s$.

The effect of stranding at low frequencies is thus to reduce these losses in the ratio $1/s$.

At high frequencies $\phi(f/R_0) = C'(f/R_0)^{\frac{1}{2}}$ and the losses are $C'(f/R_0)^{\frac{1}{2}}$ and $C's(f/sR_0)^{\frac{1}{2}} = C's^{\frac{1}{2}}(f/R_0)^{\frac{1}{2}}$, or the effect of stranding at high frequencies is to increase the losses in the ratio $s^{\frac{1}{2}}/1$.

Since in inductive coils the general field produces the main losses, LINDEMANN's results are explained.

(A). Eddy-Current Losses in a Cylinder in an Alternating Magnetic Field.

(1) The cylinder is supposed to be non-magnetic and to have electrical conductivity k. Its radius is a. The magnetic field is perpendicular to the axis of the cylinder and does not vary along the axis; otherwise its form is general. The field alternates with frequency $\omega/2\pi$, and the alternations are so slow that the dielectric current can be neglected in comparison with the conductance current. This means that the wave length of the disturbance producing the field is large compared with the dimensions of the cylinder. On the other hand, the cylinder is supposed to be long enough to render its end effects negligible.

The procedure is to represent the electric and magnetic forces by rotors* $E\epsilon^{i\omega t}$, &c. The values of these rotors are found at all points in terms of the (given) undisturbed field. Then by application of POYNTING's Theorem over unit length of the surface of the cylinder, the energy flow into the cylinder is determined.

This energy flow may be regarded as made up of two portions, one continuous and the other alternating. The former portion is the energy dissipated by eddy-currents set up in the cylinder.

(2) Take the axis of the cylinder as the axis of a right-handed system of cylindrical

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* These are the rotating vectors used to represent these quantities on the vector diagram. The term is chosen to distinguish them from the space vectors which are also involved in the problem.

co-ordinates (z, r, θ). With the assumed conditions the electromagnetic equations are

$$\left.\begin{aligned} -i\omega\mathrm{P} = \frac{1}{r}\frac{\partial \mathrm{E}}{\partial\theta}, \qquad i\omega\mathrm{Q} = \frac{\partial\mathrm{E}}{\partial r} \\ \frac{1}{r}\frac{\partial(\mathrm{Q}r)}{\partial r} - \frac{1}{r}\frac{\partial\mathrm{P}}{\partial\theta} = 4\pi k\mathrm{E} \end{aligned}\right\}, \quad \text{(1)}$$

in which P, Q represent the components of the magnetic force acting along and perpendicular to r, and E represents the electric force acting parallel to z.

Eliminating P and Q, the equation to be satisfied by E is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\mathrm{E}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\mathrm{E}}{\partial\theta^2} = 4\pi k i\omega\mathrm{E}, \quad \text{(2)}$$

the normal solution of which is

$$\mathrm{E} = \mathrm{R}_n \cos n\theta + \mathrm{S}_n \sin n\theta, \quad \text{(3)}$$

in which R_n and S_n are functions of r both satisfying the equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathrm{R}_n}{dr}\right) - \frac{n^2}{r^2}\mathrm{R}_n = 4\pi k i\omega\mathrm{R}_n. \quad \text{(4)}$$

Writing $\lambda^2 = -4\pi k i\omega$ and putting x for λr, (4) may be written

$$\vartheta^2\mathrm{R}_n + (x^2 - n^2)\,\mathrm{R}_n = 0, \quad \text{(5)}$$

in which

$$\vartheta \equiv x\frac{d}{dx}.$$

This is the general differential equation for the BESSEL functions, so that inside the cylinder the appropriate solution of (4) is

$$\mathrm{R}_n = \mathrm{A}_n\mathrm{J}_n(\lambda r), \quad \text{(6)}$$

the second solution being excluded, since the electric force is not infinite at the axis.

Outside the cylinder k is zero, so that the solution of (4) is

$$\mathrm{R}_n = \mathrm{B}_n r^n + \mathrm{C}_n/r^n \quad \text{(6A)}$$

except when $n = 0$, in which case

$$\mathrm{R}_0 = \mathrm{B}_0 \log_\epsilon r + \mathrm{C}_0. \quad \text{(6B)}$$

In order to maintain the continuity of E and $\frac{\partial\mathrm{E}}{\partial r}$ at the boundary of the cylinder, A_n, B_n, C_n must satisfy the relations

$$\left.\begin{aligned} A_nJ_n(\lambda a) &= B_na^n + C_n/a^n \\ A_n\lambda aJ'_n(\lambda a) &= n(B_na^n - C_n/a^n) \end{aligned}\right\} n \neq 0$$

$$\left.\begin{aligned} A_0J_0(\lambda a) &= B_0 \log_\epsilon a + C_0 \\ A_0\lambda aJ'_0(\lambda a) &= B_0 \end{aligned}\right\},$$

or, expressing A_n, C_n in terms of B_n and making use of the properties of the BESSEL functions,

$$\left.\begin{aligned} &\left.\begin{aligned} A_n &= 2nB_na^n/\lambda aJ_{n-1}(\lambda a) \\ C_n &= B_na^{2n}J_{n+1}(\lambda a)/J_{n-1}(\lambda a) \end{aligned}\right\} n \neq 0 \\ &A_0 = B_0/\lambda aJ'_0(\lambda a) \\ &C_0 = B_0\{J_0(\lambda a)/\lambda aJ'_0(\lambda a) - \log_\epsilon a\} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The general solution of (2) is the sum of the normal solutions of the type (3), so that the electric force may be expressed as a FOURIER series, whose form inside the cylinder is

$$E_1 = B_0\frac{J_0(\lambda r)}{\lambda aJ'_0(\lambda a)} + 2\sum_1^\infty \frac{nB_na^nJ_n(\lambda r)}{\lambda a\, J_{n-1}(\lambda a)} \cos(n\theta + \alpha_n), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and outside the cylinder is

$$E_2 = B_0\left(\log_\epsilon\frac{r}{a} + \frac{J_0(\lambda a)}{\lambda' aJ'_0(\lambda a)}\right) + \sum_1^\infty B_nr^n\left\{1 + \left(\frac{a}{r}\right)^{2n}\frac{J_{n+1}(\lambda a)}{J_{n-1}(\lambda a)}\right\}\cos(n\theta + \alpha_n). \quad . \quad (8\text{A})$$

The corresponding series for P and Q follow by differentiation using the relation (1).

The combination of the cosine and sine terms into the form cos $(n\theta + \alpha_n)$ is permissible, since the ratios of the arbitrary constants are the same for both the sine and cosine series. The values of B_n and α_n may be determined when the form of the undisturbed field is given.

(3) *Energy Dissipation in the Cylinder.*—From (8A) and (1) the values of E and Q at the surface of the cylinder are

$$E = B_0\chi_0 + \sum_1^\infty B_na^n(1 + \chi_n)\cos(n\theta + \alpha_n). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$Q = -\frac{i}{\omega a}\left\{B_0 + \sum_1^\infty nB_na^n(1 - \chi_n)\cos(n\theta + \alpha_n)\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

in which

$$\left.\begin{aligned} \chi_n &= J_{n+1}(\lambda a)/J_{n-1}(\lambda a) \\ \chi_0 &= J_0(\lambda a)/\lambda aJ'_0(\lambda a) = \tfrac{1}{4}(1 + \chi_2) - \frac{2}{\lambda^2a^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If e, q represent the instantaneous values of E and Q, the rate at which energy flows into the cylinder through a small area $d\sigma$ about the point a, θ is $eqd\sigma/4\pi$ by POYNTING'S theorem, and the integral rate of flow into unit length of the cylinder is

$$\frac{a}{4\pi}\int_0^{2\pi} eqd\sigma. \qquad (12)$$

Now, by (9) and (10), e and q have the forms $\Sigma u_n \cos(n\theta + \alpha_n)$ and $\Sigma v_n \cos(n\theta + \alpha_n)$ respectively.

Using these forms in (12), and remembering that

$$\int_0^{2\pi} \cos(n\theta+\alpha_n)\cos(m\theta+\alpha_n)\,d\theta = 0, \qquad n \neq m,$$

the integral rate of flow becomes

$$\frac{a}{4}(2u_0v_0+\Sigma u_nv_n). \qquad (13)$$

To determine the products u_nv_n, the obvious method is to determine the real parts of the complex coefficients in (9) and (10). The product u_nv_n expressed as a function of time would then be of the form $w_n \cos \omega t \cos(\omega t + \phi_n)$, so that the rate of energy dissipation in the cylinder would be $\frac{1}{2}w_n \cos \phi_n$.

A better method is to make use of the method of conjugates. If U′, V′ are the conjugates* of two complex quantities U, V; u, v their real parts, then

$$u = \tfrac{1}{2}(\mathrm{U}+\mathrm{U}'), \qquad v = \tfrac{1}{2}(\mathrm{V}+\mathrm{V}'),$$

so that

$$uv = \tfrac{1}{4}(\mathrm{U}+\mathrm{U}')(\mathrm{V}+\mathrm{V}').$$

Further, if U, V rotate with time in the same sense, U′, V′ will rotate in the opposite sense, so that UV′ and VU′ will not rotate. The steady portion of uv is thus

$$\tfrac{1}{4}(\mathrm{UV}'+\mathrm{U}'\mathrm{V}).$$

Applying this to the present case, the steady flow into unit length of the cylinder is from (9), (10) and (13)

$$\mathrm{W} = \frac{i}{8\omega}\left\{\mathrm{B}_0\mathrm{B}'_0(\chi_0-\chi'_0)+\sum_1^\infty na^{2n}\mathrm{B}_n\mathrm{B}'_n(\chi_n-\chi'_n)\right\},$$

in which as before the accents denote conjugates.

Now χ_n is a function of λa and $\lambda^2 = +4\pi ki\omega$, so that, putting

$$z^2 = 4\pi k\omega a^2, \qquad (14)$$

χ_n may be written

$$\chi_n = \phi_n(z)-i\psi_n(z), \qquad (15)$$

from which

$$i(\chi_n-\chi'_n) = 2\psi_n(z),$$

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* If $\mathrm{U} = \mathrm{A}\epsilon^{i\psi}$, then $\mathrm{U}' = \mathrm{A}\epsilon^{-i\psi}$ is the conjugate of U.

and when $n = 0$

$$i(\chi_0-\chi'_0) = \frac{4}{z^2}+\tfrac{1}{2}\psi^2(z).$$

Using these expressions in the equation for W with (B_n) for the modulus of B_n,

$$W = \frac{1}{4\omega}\left[(B_0)^2\left\{\frac{2}{z^2}+\tfrac{1}{4}\psi_2(z)\right\}+\sum_1^\infty n(B_n)^2a^{2n}\psi_n(z)\right]. \quad . \quad . \quad . \quad . \quad (16)$$

The energy dissipation in unit length of the cylinder is thus expressed in terms of coefficients B_n depending on the form of the applied magnetic field and of functions ψ_n having argument $z = 2a(\pi k\omega)^{\frac{1}{2}}$. The functions ϕ_n, ψ_n are discussed in the next section. As regards the coefficients B_n, if the components of the magnetic force in the undisturbed field are P_0, Q_0 these components may be expressed in the forms

$$\left.\begin{aligned} P_0 &= \sum_1^\infty K_n r^{n-1}\sin(n\theta+\alpha_n) \\ Q_0 &= \frac{K_0}{r}+\sum_1^\infty K_n r^{n-1}\cos(n\theta+\alpha_n) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

at all points outside the cylinder as these expressions are derivatives of a potential function satisfying LAPLACE'S equation and constant along the axis of the cylinder.

Further, by differentiation of (8A), similar expressions to (17) are obtained, when λ is made zero—that is, when the disturbance due to eddy-currents in the cylinder is removed. These expressions are identical with (17) if we make

$$B_0 = i\omega K_0, \qquad B_n = i\omega K_n/n.$$

Hence, using K_n in place of B_n in (16)

$$W = \tfrac{1}{4}\omega\left[(K_0)^2\left\{\frac{2}{z^2}+\tfrac{1}{4}\psi_2(z)\right\}+\sum_1^\infty (K_n)^2a^{2n}\psi_n(z)/n\right]. \quad . \quad . \quad . \quad . \quad (18)$$

(4) *The Functions* ϕ_n *and* ψ_n.—These functions are defined by

$$\phi_n(z)-i\psi_n(z) = J_{n+1}(\sqrt{-i}z)/J_{n-1}(\sqrt{-i}z).$$

Series formulæ for these functions have been developed by the author.*

The cases $n = 1$ and $n = 2$ are the most important ones, and in these cases ϕ and ψ may be expressed in terms of ber and bei functions as follows :—

Let

$$\left.\begin{aligned} X(z) &= \mathrm{ber}^2 z+\mathrm{bei}^2 z \\ V(z) &= \mathrm{ber}'^2 z+\mathrm{bei}'^2 z \\ Z(z) &= \mathrm{ber}\, z\, \mathrm{ber}' z+\mathrm{bei}\, z\, \mathrm{bei}' z \\ W(z) &= \mathrm{ber}\, z\, \mathrm{bei}' z-\mathrm{bei}\, z\, \mathrm{ber}' z \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

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* BUTTERWORTH, 'Proc. Phys. Soc. Lond.,' vol. XXV, p. 294, 1913.

Then

$$\left.\begin{aligned} \phi_1(z) &= \frac{2}{z}\frac{\mathrm{W}(z)}{\mathrm{X}(z)}-1, \qquad & \phi_2(z) &= \frac{4}{z}\frac{\mathrm{Z}(z)}{\mathrm{V}(z)}-1 \\ \psi_1(z) &= \frac{2}{z}\frac{\mathrm{Z}(z)}{\mathrm{X}(z)}, & \psi_2(z) &= \frac{4}{z}\frac{\mathrm{W}(z)}{\mathrm{V}(z)}-\frac{8}{z^2} \end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (20)$$

The combinations W/X, Z/X, W/V and Z/V are tabulated.*

In the limiting cases of z very small or very large, it may be shown from the formulæ already cited that ϕ_n and ψ_n assume the following simple forms:—

$$z \text{ small} \quad . \; . \; . \left\{\begin{aligned} \phi_n &= -2z^4/(2n)^2(2n+2)(2n+4) \\ \psi_n &= z^2/2n(2n+2) \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (21\text{A})$$

$$z \text{ large} \; . \; . \; . \; . \; . \quad \phi_n = -1, \qquad \psi_n = 2n/\sqrt{2}z. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (21\text{B})$$

In regard to the limitations of (21A), the following table of values of ψ_1, ψ_2 (the functions most generally used) has been calculated:—

z.	ψ_1.	ψ_2.	ψ_1/z^2.	ψ_2/z^2.
0·0	0·0000	0·0000	0·1250	0·04167
0·5	0·03119	0·01041	0·1248	0·04164
1·0	0·1215	0·04149	0·1215	0·04149
1·5	0·2458	0·0918	—	—
2·0	0·3448	0·1563	0·0862	0·03908
2·5	0·3770	0·2244	—	—
3·0	0·3600	0·2827	0·0400	0·03141
3·5	0·3257	0·3212	—	—
4·0	0·2920	0·3389	—	—
4·5	0·2643	0·3408	—	—
5·0	0·2416	0·3337	—	—

For ψ_1, (21A) is a good approximation up to $z = 0·5$ and a fair approximation up to $z = 1$. For higher values of n the range of (21A) increases.

In regard to (21B), its region of application has not been reached at $z = 5$, but if we take a second approximation we find, when z is large,

$$z^2\psi_1 = \sqrt{2}z-1, \qquad z^2\psi_2 = 2\sqrt{2}z-6. \quad . \; . \; . \; . \; . \; . \; . \quad (21\text{C})$$

These formulæ give the following values for ψ_1, ψ_2:—

$z =$	2	3	4	5
$\psi_1 =$	0·457	0·360	0·291	0·243
$\psi_2 =$	−0·086	0·276	0·332	0·326

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* SAVIDGE, 'Phil. Mag.,' 6, 19, p. 49, 1910. ROSA and GROVER, 'Bull. Bureau of Stands.,' p. 226, 1912.

while the actual values are

$$\psi_1 = \quad 0\cdot345 \qquad 0\cdot360 \qquad 0\cdot292 \qquad 0\cdot242$$
$$\psi_2 = \quad 0\cdot156 \qquad 0\cdot287 \qquad 0\cdot339 \qquad 0\cdot334$$

so that if z is greater than 3 the values given by (21c) are fair approximations to the true values. The values of ψ_1 and ψ_2 from $z = 0$ to $z = 5$ are plotted in fig. 1.

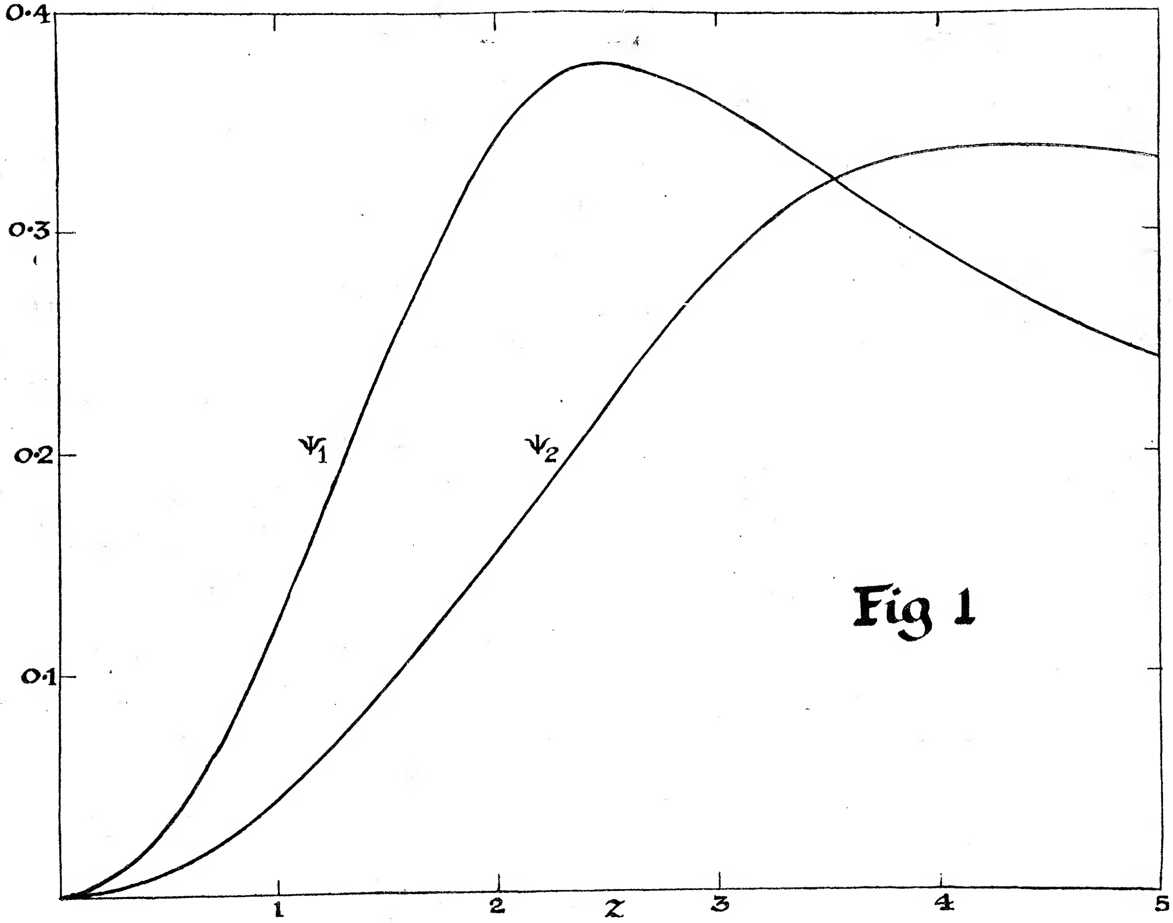


Fig 1

(5) *Eddy-Current Losses at Low Frequencies.*—The argument z is related to the radius and conductance of the cylinder and the frequency of alternation of the field by the formula

$$z^2 = 4\pi k\omega a^2.$$

If R_0 is the electrical resistance per unit length of the cylinder, the formula may be written

$$z^2 = 4\omega/R_0. \qquad (22)$$

The frequency $w/2\pi$ will be defined to be *low* when z is less than unity, so that the condition of low frequency is $\omega < R_0/4$. When this holds, formula (21A) applies in regard to ψ, so that by (18) the rate of dissipation of energy is

$$W = \tfrac{1}{8} R_0 (K_0)^2 + \frac{\omega^2}{R_0}\left\{\tfrac{1}{96}(K_0)^2 + \sum_1^\infty (K_n)^2 a^{2n}/2n^2(2n+2)\right\}. \quad . \quad . \quad . \quad (23)$$

Now by (17) the terms involving K_0 are due to a field whose components outside the cylinder are $Q_0 = K_0/r$, $P_0 = 0$. This field can only be due to a current of magnitude $I = \tfrac{1}{2}K_0$ distributed symmetrically round the axis and flowing parallel to the axis.

Hence the energy dissipation due to such a current is

$$W_1 = \tfrac{1}{2}R_0\left(1 + \tfrac{1}{12}\frac{\omega^2}{R_0^2}\right)I^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

This is the usual formula for the skin effect at low frequencies.

If a uniform field H is acting on the cylinder, then $H = K_1$, $K_2 = K_3 = \ldots = 0$, so that the energy dissipation due to a uniform field H is

$$W_2 = \tfrac{1}{8}\omega^2 H^2 a^2/R_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The remaining terms are due to non-uniformity of the field.

If the external field is expressed in a FOURIER series of the form (17), and if the coefficient of the term $\cos(n\theta + \alpha_n)$ in the series for Q_0 has the value L_n at the surface of the cylinder, then this portion of the field contributes an amount

$$\omega^2 a^2 L^2_n/2n^2(2n+2)R_0$$

to the energy dissipation.

The way in which n occurs in this expression shows how unimportant are the higher terms of the FOURIER series in producing eddy losses at low frequencies.

The assumption that the external field is uniform and has its central value will, therefore, in most cases give a good approximation to the actual loss when the frequency is low. In illustration, suppose the external field to be due to a thin wire carrying current I, and stretched parallel to the cylinder at a distance D from the axis. The value of Q_0 in the plane common to the axis and the wire is $2I/(D - r)$, or, in ascending powers of r,

$$\frac{2I}{D}\left(1 + \frac{r}{D} + \frac{r^2}{D^2} + \ldots\right),$$

so that

$$L_n = \frac{2I}{D}\cdot\frac{a^n}{D^n},$$

and therefore the energy dissipation is

$$\frac{\omega^2 a^2 I^2}{R_0 D^2}\left(\frac{1}{1^2 2} + \frac{1}{2^2 3}\frac{a^2}{D^2} + \frac{1}{3^2 4}\frac{a^4}{D^4} + \ldots\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In the extreme case in which the wire touches the cylinder, the sum of the series becomes

$$\frac{\pi^2}{6}-1 = 0{\cdot}64493.$$

Upon the assumption of a uniform field, the first term of the series would be the only one employed; and as this is 1/2, the correction due to non-uniformity of the field is a multiplying factor ranging from 1·00 to 1·29.

If there are two thin parallel wires and the cylinder is situated symmetrically between them, the axes of wires and cylinder being coplanar, the alternate terms of the series (26) vanish, and the losses become

$$\frac{4\omega^2 a^2 \mathrm{I}^2}{\mathrm{R}_0 \mathrm{D}^2}\left(\frac{1}{1^2 2}+\frac{1}{3^2 4}\frac{a^4}{\mathrm{D}^4}+\frac{1}{5^2 6}\frac{a^8}{\mathrm{D}^8}+\ldots\right) \quad \ldots \ldots \quad (27)$$

if the currents flow in opposite directions in the two wires, and

$$\frac{4\omega^2 a^2 \mathrm{I}^2}{\mathrm{R}_0 \mathrm{D}^2}\left(\frac{1}{2^2 3}\frac{a^2}{\mathrm{D}^2}+\frac{1}{4^2 5}\frac{a^6}{\mathrm{D}^6}+\ldots\right) \quad \ldots \ldots \quad (28)$$

when the currents flow in the same direction.

When the wires touch the cylinder, (27) reduces to

$$\frac{4\omega^2 \mathrm{I}^2}{\mathrm{R}_0}\left(\frac{\pi^2}{8}-\log_\epsilon 2\right) = \frac{4\omega^2 \mathrm{I}^2}{\mathrm{R}_0} \times 0{\cdot}54055$$

and (28) to

$$\frac{4\omega^2 \mathrm{I}^2}{\mathrm{R}_0}\left(\frac{\pi^2}{24}+\log_\epsilon 2-1\right) = \frac{4\omega^2 \mathrm{I}^2}{\mathrm{R}_0} \times 0{\cdot}10438.$$

The uniform field theory would give $\frac{4\omega^2 \mathrm{I}^2}{\mathrm{R}_0} \times 0{\cdot}5000$ and zero respectively for these cases.

(6) *Eddy-Current Losses at High Frequencies.*—At very high frequencies

$$\psi_n = 2n/\sqrt{2}z = n\sqrt{\mathrm{R}_0/2\omega},$$

so that by (18) the energy dissipation is

$$\mathrm{W} = \tfrac{1}{4}\sqrt{\mathrm{R}_0\omega/2}\left\{\tfrac{1}{2}(\mathrm{K}_0)^2+\sum_1^\infty (\mathrm{K}_n)^2 a^{2n}\right\}. \quad \ldots \ldots \quad (29)$$

The first term is due to a current $\mathrm{I} = \frac{1}{2}\mathrm{K}_0$ distributed symmetrically round the axis of the cylinder and flowing in a direction parallel to the axis, and when this is the only factor producing the field the energy dissipation is

$$\mathrm{W}_1 = \tfrac{1}{2}\sqrt{\mathrm{R}_0\omega/2}\,\mathrm{I}^2. \quad \ldots \ldots \quad (30)$$

This is the formula for the skin effect at high frequencies.

The energy dissipation due to a uniform field H is got by putting $K_1 = H$, $K_2 = K_3 = \ldots = 0$, giving

$$W_2 = \tfrac{1}{4}H^2a^2\sqrt{R_0\omega/2}. \qquad (31)$$

If the field is non-uniform, a comparison of (29) with (17) shows that the various harmonic terms in the field produce terms of equal importance in the expression for the eddy-current losses.

The examples of the last section give, for the single thin wire,

$$W = \sqrt{R_0\omega/2}\,\frac{a^2I^2}{D^2}\left(1+\frac{a^2}{D^2}+\frac{a^4}{D^4}+\ldots\right) = \frac{a^2I^2}{D^2-a^2}\times\sqrt{R_0\omega/2}, \qquad (32)$$

and for the pair of wires,

$$W = \frac{a^2D^2}{D^4-a^4}\,I^2\sqrt{R_0\omega/2} \quad \text{or} \quad \frac{a^4}{D^4-a^4}\,I^2\sqrt{R_0\omega/2}, \qquad (33)$$

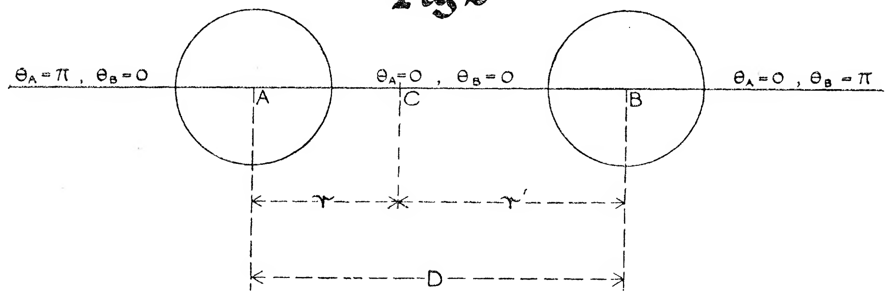
according as the currents flow in opposite or the same direction in the two wires.

As regards (32), it is seen that the uniform field theory may be applied if we take as the uniform field the field at the point where the tangent plane through the wire touches the surface of the cylinder.

(B). Eddy-Current Losses in Two Parallel Cylinders Carrying Equal Currents.

(7) If the field acting on the cylinder is due to currents in neighbouring cylinders, then, because of the distortion of the current distribution in these cylinders, the external field acting on the cylinder under consideration is itself variable with frequency, and the assumption that this field is that which would occur if *all* eddy-currents are absent will lead to wrong results. The case of two similar parallel cylinders carrying equal currents may be solved by considerations of symmetry.

Fig 2


$\theta_A = \pi$, $\theta_B = 0$
$\theta_A = 0$, $\theta_B = 0$
$\theta_A = 0$, $\theta_B = \pi$
A
C
B
r
r'
D

(8) Let the cylinders each have radius a, and let the distance of their centres be D. Take two systems of cylindrical co-ordinates (fig. 2), the first system (r, θ_A) having A as origin and AB as the line of zero θ, and the second system (r', θ_B) having B as

origin and BA as the line of zero θ. Consider the field at the point C external to the cylinders situated on AB and due to two current systems flowing in the two cylinders parallel to the axes and symmetrical on either side of AB. By symmetry α_n of equation (17) is zero; and since θ is zero, we have from (10), using K_n instead of B_n, for the first system of co-ordinates,

$$Q = \frac{K_0}{r} + \sum_1^\infty K_n r^{n-1}\left\{1-\left(\frac{a}{r}\right)^{2n}\chi_n\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

for the second system

$$Q = \frac{K'_0}{r'} + \sum_1^\infty K'_n r'^{n-1}\left\{1-\left(\frac{a}{r'}\right)^{2n}\chi_n\right\}. \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Also, if the currents are equal and *similarly* directed in the two cylinders,

$$K'n = -Kn;$$

if *oppositely* directed,

$$K'n = +Kn.$$

Further, (34) may be divided into two portions,

$$Q_1 = \frac{K_0}{r} - \sum_1^\infty K_n \frac{a^{2n}}{r^{n+1}}\chi_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

and

$$Q_2 = \sum_1^\infty K_n r^{n-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

the former arising from causes *inside* the cylinder A, and the latter from causes *outside* A, which in this case are located *inside* B, and therefore Q_2 has also the value

$$Q_2 = \frac{K'_0}{r'} - \sum_1^\infty K'_n \frac{a^{2n}}{r'^{n+1}}\chi_n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Equating (37) and (38),

$$\sum_1^\infty K_n r^{n-1} = \frac{K'_0}{r'} - \sum_1^\infty K'_n \frac{a^{2n}}{r'^{n+1}}\chi_n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Putting $K'n = \pm Kn$, $r' = D-r$, expanding the right-hand side of (39) in ascending powers of r, and equating coefficients of r^{n-1}, a series of equations are obtained to determine K_n in terms of K_0. Now if I is the total current in either wire, $K_0 = 2I$, so that the method yields the values of K_n completely. Thus, in the case where the currents are similarly directed, $K'n = -Kn$; and on equating coefficients we find

$$\begin{aligned}
K_1 a &= -2I\mu + (K_1 a)\,\mu^2\chi_1 + (K_2 a^2)\,\mu^3\chi_2 + \dots \\
K_2 a^2 &= -2I\mu^2 + 2\,(K_1 a)\,\mu^3\chi_1 + 3\,(K_2 a^2)\,\mu^4\chi_2 + \dots \\
K_3 a^3 &= -2I\mu^3 + 3\,(K_1 a)\,\mu^4\chi_1 + 6\,(K_2 a^2)\,\mu^5\chi_2 + \dots,
\end{aligned}$$

in which $\mu \equiv a/D$ and is less than $1/2$.

Solving by successive approximations to the order μ^4,

$$\left.\begin{aligned} K_1 a &= -2I\mu\,\{1+\mu^2\chi_1+\mu^4(\chi_1^2+\chi_2)+\dots\} \\ K_2 a^2 &= -2I\mu^2(1+2\mu^2\chi_1+\dots) \\ K_3 a^3 &= -2I\mu^3(1+\dots) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

In the expression for the eddy-current losses (18) the moduli of the complex quantities K_1, K_2, K_3 are required, which, from (40) with $\chi_n = \phi_n - i\psi_n$, are

$$\left.\begin{aligned} (K_1)^2 &= \frac{4I^2}{D^2}\,\{1+2\mu^2\phi_1+\mu^4(2\phi_2+3\phi_1^2-\psi_1^2)+\dots \\ (K_2)^2 &= \frac{4I^2}{D^4}(1+4\mu^2\phi_1+\dots) \\ (K_3)^2 &= \frac{4I^2}{D^6}(1+\dots) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Substituting in (18), the energy dissipation per unit length in either cylinder is given by

$$W = \omega I^2\left[\frac{2}{z^2}+\tfrac{1}{4}\psi_2+\mu^2\psi_1\left\{1+\mu^2\left(2\phi_1+\tfrac{1}{2}\frac{\psi_2}{\psi_1}\right)+\mu^4\left(2\phi_2+\tfrac{1}{3}\frac{\psi_3}{\psi_1}+3\phi_1^2-\psi_1^2+\frac{2\phi_1\psi_2}{\psi_1}\right)\right\}\right] \ . \quad (42)$$

When the currents are in opposite directions a similar treatment gives

$$W = \omega I^2\left[\frac{2}{z^2}+\tfrac{1}{4}\psi_2+\mu^2\psi_1\left\{1+\mu^2\left(\tfrac{1}{2}\frac{\psi_2}{\psi_1}-2\phi_1\right)+\mu^4\left(3\phi_1^2-\psi_1^2-2\phi_2-\frac{2\phi_1\psi_2}{\psi_1}+\tfrac{1}{3}\frac{\psi_3}{\psi_1}\right)\right\}\right] \ . \quad (43)$$

In (42) and (43) the term

$$\omega I^2\left(\frac{2}{z^2}+\tfrac{1}{4}\psi_2\right)$$

is due to the ordinary skin effect. Since $2\omega/z^2 = \frac{1}{2}R_0$, this term may be written $\frac{1}{2}R_0 I^2\{1+F(z)\}$ in which $F(z)\ \{\equiv \frac{1}{8}z^2\psi_2\}$ is plotted in fig. 3 up to $z = 5$. When z is greater than 5, then, by (21c),

$$F(z) \doteqdot (\sqrt{2}z-3)/4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

This is shown in fig. 3 by the broken line A.

The next term, $\mu^2\psi_1$, is due to the proximity of the two cylinders when they are so far apart that the quantity in { } may be regarded as practically unity. It would have been the term obtained by assuming the current in the second cylinder as concentrated on the axis and producing a uniform field of strength 2 I/D on the first cylinder. Including the proximity effect to this order we may write

$$W = \tfrac{1}{2}R_0 I^2\left\{1+F(z)+\frac{d^2}{D^2}G(z)\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

in which d ($\equiv 2a$) is the diameter of either cylinder, and $G(z) \equiv \frac{1}{8}z^2\psi_1$ is plotted in fig. 3, its limiting value

$$G(z) \doteqdot (\sqrt{2}z-1)/8 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

being indicated by the broken line (B).

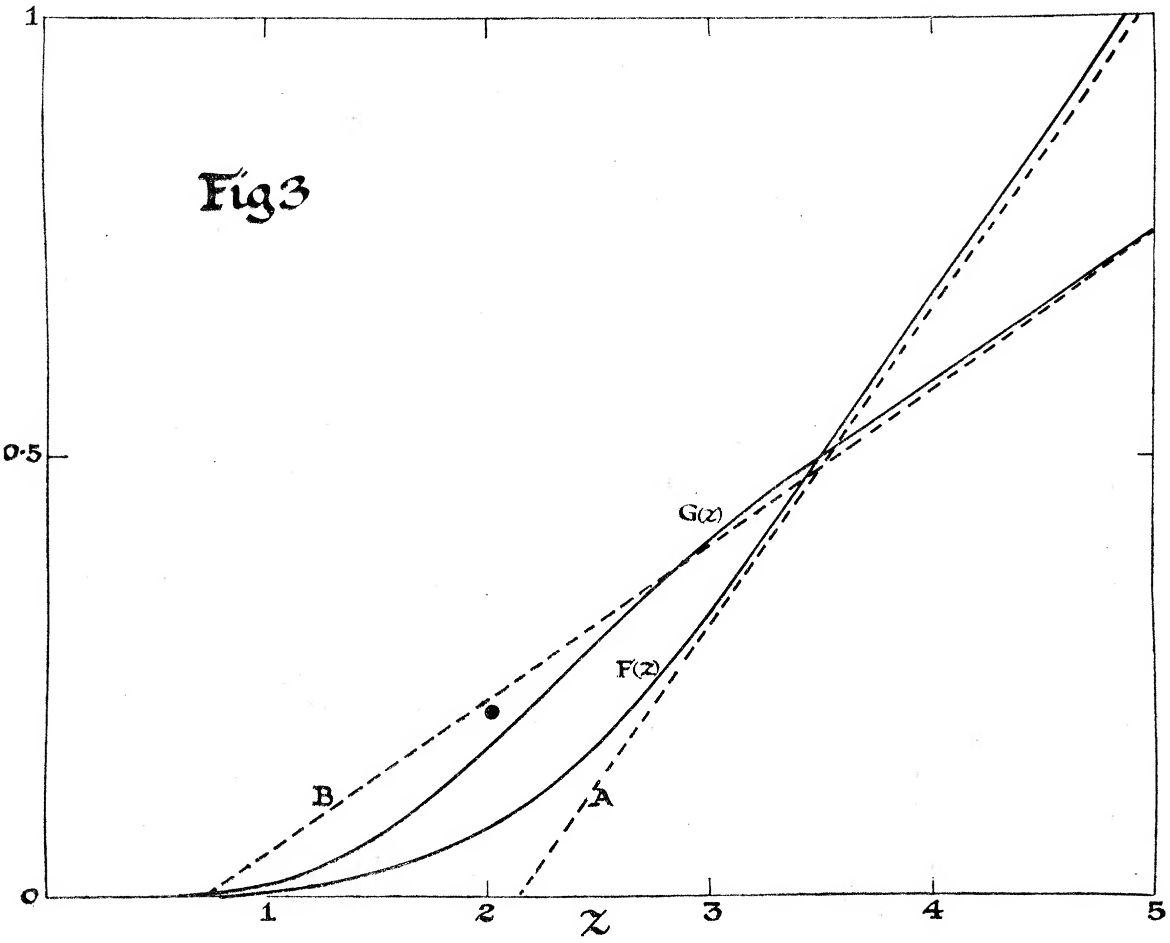


Fig3

Finally the expression included in { } includes the effect both of disturbance of current distribution due to proximity and of non-uniformity of the field.

At low frequencies, by (21A), ϕ_n is negligible and both (42) and (43) give for { }

$$1+\tfrac{1}{6}\mu^2+\tfrac{1}{18}\mu^4+\ldots .$$

This is identical with the result obtained for the thin wire (Equation 26), so that at low frequencies the distortion of distribution due to reaction of eddy-currents is quite negligible, and the effect of non-uniformity of the field will give in the extreme case where the cylinders are touching a correcting factor of amount 1·0456 to be applied to the result obtained by the uniform field theory.

At frequencies so high that (21B) holds, the factor { } becomes

$$1-\mu^2-2\mu^4$$

when the currents are similarly directed, and

$$1+3\mu^2+10\mu^4$$

when the currents are oppositely directed; while, if distortion is neglected, its value is

$$1+\mu^2+\mu^4$$

by (32). Thus the effects of distortion and of non-uniformity of field are equally important. When the currents are similarly directed the distortion is such as to tend to reduce the losses, and when the currents are oppositely directed the losses are increased.

The term involving μ^4 will contribute less than 1 per cent. at high frequencies if $D > 2 \cdot 5d$ with the currents oppositely directed, and if $D > 2d$ with the currents similarly directed.

To this accuracy we may write for any frequency

$$W = \tfrac{1}{2}R_0 \left\{1+F(z)+G(z)\frac{d^2}{D^2}\left(1+H(z)\frac{d^2}{D^2}\right)\right\} I^2, \quad . \quad . \quad . \quad . \quad . \quad (46)$$

in which

$$H(z) = \tfrac{1}{4}\left(\tfrac{1}{2}\frac{\psi_2}{\psi_1} \pm 2\phi_1\right).$$

If we assume the remaining terms to be in geometrical progression (46) may be written

$$W = \tfrac{1}{2}R_0 \left\{1+F(z)+\frac{G(z)}{1-\dfrac{d^2}{D^2}H(z)}\frac{d^2}{D^2}\right\} I^2. \quad . \quad . \quad . \quad . \quad . \quad (47)$$

At low frequencies this formula gives $1 \cdot 043$ as the correction for non-uniformity when the cylinders touch, and will certainly hold to 1 per cent. up to $D = 2d$ at extremely high frequencies.

$H(z)$ is plotted in fig. 4, up to $z = 5$, the curve I holding when the currents are oppositely directed, and II when the currents are similarly directed.

Since the effective resistance R' of a coil system is such that

$$W = \tfrac{1}{2}R'I^2,$$

the effective resistance (apart from electrostatic capacities) per unit length of a pair of parallel wires is given by the formula*

$$R' = R_0 \left\{1+F(z)+\frac{G(z)}{1-\dfrac{d^2}{D^2}H(z)}\frac{d^2}{D^2}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

in which d is the diameter of either wire D the distance of their centres, and F, G, H are functions of $z \{\equiv 2\sqrt{\omega/R_0}\}$ drawn in fig. 3 and 4, and tabulated below.

---

* At extremely high frequencies it may be shown that ratio of the resistance of a go-and-return system to the skin resistance is given by $D/\sqrt{D^2-d^2}$. Formula (48) is then 3 per cent. in error when $d = 0 \cdot 8D$.

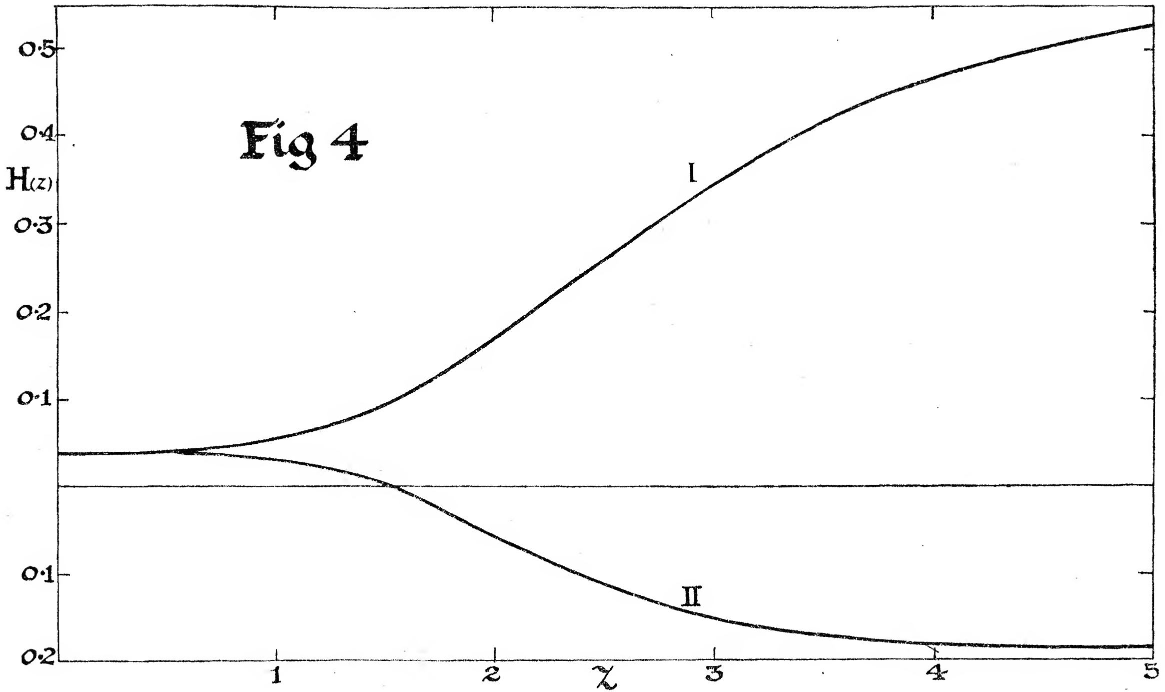


Fig 4

VALUES of F, G, H in Formula (48).

$$z = 2\sqrt{\omega/R_0}\begin{cases}\omega/2\pi = \text{frequency.}\\ R_0 = \text{D.C. resistance of cylinder per centimetre.}\end{cases}$$

z.	F (z).	G (z).	H (z).	
			I. Currents Opposite.	II. Currents Similar.
0·0	0·000	0·000	0·0417	+0·0417
0·5	0·000326	0·000975	0·042	+0·041
1·0	0·00519	0·01519	0·053	+0·033
1·5	0·0258	0·0691	0·092	+0·001
2·0	0·0782	0·1724	0·169	−0·056
2·5	0·1756	0·295	0·263	−0·114
3·0	0·318	0·405	0·348	−0·152
3·5	0·492	0·499	0·416	−0·170
4·0	0·678	0·584	0·466	−0·176
4·5	0·862	0·669	0·503	−0·181
5·0	1·042	0·755	0·530	−0·185
Large	$(\sqrt{2}z-3)/4$.	$(\sqrt{2}z-1)/8$	0·750	−0·250

(9) *Test of Formula* (48) *by Comparison with Experimental Results.*—An extensive series of measurements of the resistance of a go-and-return system of parallel conductors has been made by KENNELLY, LAWS and PIERCE.*

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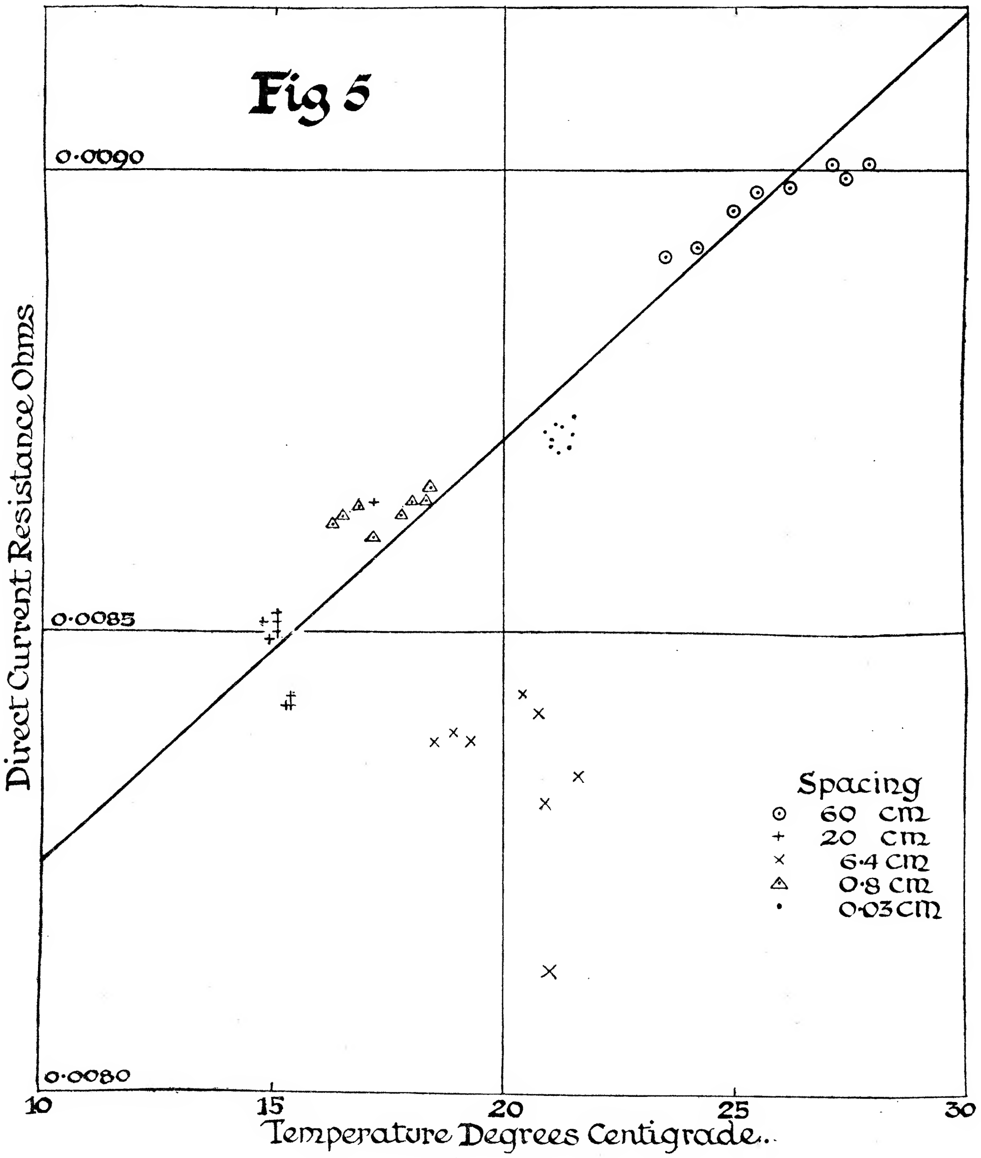
* ' Trans. American Inst. El. Eng.,' Vol 35, Part 2, 1953, 1915.

The wire used was copper of diameter 1·168 cm., and the frequencies employed ranged from 60 to 5,000 cycles per second, so that the range of z in formula (48) is from 0·84 to 7·7. The spacing (D − d) varied from 0·03 cm. to 60 cm., so that the observations afford a very complete check on the adequacy of the formula.

The resistance of the loop, which for direct currents was of the order 0·01 ohm, was measured by an alternating current bridge. In this method variation of contact resistance would probably be the chief source of trouble. As to whether this effect is appreciable, may be judged by a comparison of the tabulated direct current resistances after allowing for temperature variations. In fig. 5 this is done by plotting the resistances on a resistance-temperature diagram. Of the five groups of observations at spacings 60 cm., 20 cm., 6·4 cm., 0·8 cm., 0·03 cm., four show to within half a per

Fig 5



cent. a linear increase of resistance with temperature, this increase agreeing with the temperature coefficient of copper. The fifth group, that corresponding to the spacing 6·4 cm., is low. The discrepancy is removed if we suppose the measured resistance in error by 4 per cent. (0·0004 ohm).

In the following tables the ratio of alternating current resistance R′ to the direct current resistance R has been calculated from formula (48) and compared with the observed value. This ratio is not independent of the temperature as the eddy-current resistance varies with temperature in a different way to that of the direct current resistance. The value of z has been calculated using that value of R which corresponds to the temperature of the observation as deduced from fig. 5.

Tables comparing observed values of the effective resistance of two parallel wires with the calculated values :—

θ = temperature of observation.

f = frequency in cycles per second.

R = direct current resistance.

$R_S - R$ = increase in resistance due to skin effect.

R_P = increase in resistance due to proximity of wires.

$R' = R_S + R_P$ = total alternating current resistance.

Table I.—Spacing = 60 cm.

θ° C.	23·5	24·2	25·0	25·6	26·2	27·1	27·4	27·9
f	60	306	888	1600	2040	3065	3950	5000
Calculated R_S/R	1·0047	1·108	1·560	2·045	2·270	2·708	3·030	3·372
Calculated R_P/R	Negligible	—	—	—	—	—	—	—
Calculated R'/R	1·0047	1·108	1·560	2·045	2·270	2·708	3·030	3·372
Observed R'/R	1·0038	1·111	1·587	2·042	2·279	2·694	3·034	3·361
Difference per cent.	+0·1	−0·2	−1·4	+0·2	−0·4	+0·5	−0·1	+0·3

Table II.—Spacing = 20 cm.

θ° C.	17·2	15·2	15·2	15·0	14·9	15·2	15·4	15·3	15·4
f	60	288	868	1663	2061	3063	3112	3860	5040
Calculated R_S/R	1·0048	1·106	1·578	2·120	2·328	2·775	2·790	3·075	3·472
Calculated R_P/R	0·00004	0·0006	0·002	0·002	0·003	0·003	0·004	0·004	0·004
Calculated R'/R	1·0048	1·107	1·580	2·122	2·331	2·778	2·799	3·079	3·476
Observed R'/R	1·0058	1·106	1·584	2·120	2·313	2·755	2·781	3·067	3·446
Difference per cent.	−0·1	+0·1	−0·3	+0·1	+0·8	+0·8	+0·5	+0·4	+0·9

TABLE III.—Spacing = 6·4 cm.

θ° C.	18·5	18·9	19·3	20·4	20·7	20·9	21·0	21·0	21·6	—
f	60	266	582	923	1465	2019	1992	3028	3960	5320
Calculated										
R_S/R ...	1·0048	1·088	1·340	1·602	1·984	2·280	2·268	2·728	3·075	3·470
R_P/R ...	0·0003	0·0045	0·0099	0·0132	0·017	0·021	0·020	0·026	0·031	0·036
R'/R ...	1·0051	1·093	1·350	1·615	2·001	2·301	2·288	2·754	3·106	3·506
Observed R'/R	1·0087	1·100	1·354	1·640	2·037	2·344	2·322	2·851	3·145	3·558
Difference per cent.	−0·4	−0·6	−0·3	−1·6	−1·8	−1·9	−1·5	−3·6	−1·2	−1·5

This group is abnormal on the resistance-temperature diagram. If we assume R′ and R as measured are *both* too small by a constant amount = 0·04 R, the group becomes normal on the resistance-temperature diagram and gives the following values replacing the observed R′/R:—

Corrd. Obsd. R'/R ...	1·008	1·096	1·343	1·617	2·000	2·296	2·275	2·784	3·062	3·466
Difference per cent. ...	−0·3	−0·3	+0·5	−0·2	+0·0	+0·2	+0·6	−1·1	+1·5	+1·5

TABLE IV.—Spacing = 0·8 cm.

θ° C. ...	—	—	—	16·3	16·5	16·9	17·2	17·8	18·0	18·3	18·4
f... ...	60	239	671	1068	1509	1991	1988	2486	3028	3880	4900
Calculated											
R_S/R ...	1·0050	1·073	1·424	1·732	2·028	2·283	2·280	2·517	2·744	3·067	3·409
R_P/R ...	0·0052	0·061	0·187	0·254	0·314	0·382	0·380	0·424	0·477	0·550	0·620
R'/R ...	1·0102	1·134	1·611	1·986	2·342	2·665	2·660	2·941	3·221	3·617	4·029
Observed R'/R ...	1·0124	1·132	1·604	1·981	2·330	2·643	2·638	2·912	3·179	3·587	3·955
Difference per cent.	−0·2	+0·2	+0·4	+0·3	+0·5	+0·8	+0·8	+1·0	+1·3	+0·8	+1·8

The calculated values are in general too high. A spacing 0·85 cm. would give the following calculated values and differences:—

R'/R ...	1·0100	1·131	1·602	1·973	2·326	2·646	2·641	2·920	3·197	3·589	3·998
Difference per cent.	−0·3	−0·1	−0·1	−0·4	−0·2	+0·1	+0·1	+0·3	+0·6	+0·1	+0·4

Table V.—Spacing = 0·03 cm.

$\theta°$ C.	21·1	21·4	21·5	21·5	21·2	21·0	20·9	21·0	21·1
f	60	236	740	1000	1473	2038	3058	3918	5170
Calculated R_S/R	1·005	1·068	1·464	1·658	1·995	2·31	2·74	3·06	3·46
Calculated R_P/R	0·015	0·165	0·759	1·00	1·37	1·83	2·56	3·09	3·72
Calculated R'/R	1·020	1·243	2·223	2·66	3·37	4·14	5·30	6·15	7·18
Observed R'/R	1·017	1·244	2·231	2·688	3·460	4·272	5·522	6·449	7·512
Difference per cent. ...	+0·3	−0·1	−0·4	−1·1	−3·2	−3·5	−4·3	−5·0	−4·5

The calculated values are in general too low, but with so small a spacing R′ is varying rapidly. The formula gives the following values of R′/R when the wires touch:—

R'/R	1·021	1·253	2·288	2·75	3·52	4·35	5·64	6·57	7·71
Difference per cent. ...	+0·4	+0·7	+2·6	+2·2	+1·7	+1·4	+1·8	+1·6	+2·7

In Tables I. and II. the skin effect is the only one of importance and very good agreement is obtained. These tables really check the experimental observations as the skin effect formula is well established. Tables IV. and V. form the real test of the proximity effect. It is seen that the small discrepancies are sufficiently accounted for by a slight adjustment (0·5 mm. at most) in the spacing. In Table III. the skin effect is predominant, but there is a rather large discrepancy. It is noteworthy that this group also shows a discrepancy on the resistance-temperature diagram, and that both the discrepancies are removed if we assume the measured values of R′ and R to be both in error by 0·04 R.

(C) Losses in Parallel Wire Systems and in Short Coils.

(10) When the field acting upon the cylinder is uniform and has magnitude H, then by (18) the eddy-current losses per unit of length are given by

$$W = \tfrac{1}{4}\omega a^2 H^2 \psi_1(z),$$

or eliminating ω by $\omega = z^2 R_0/4$, and putting $\frac{1}{8} z^2 \psi_1 = G(z)$, $2a = d$,

$$W = \tfrac{1}{8} R_0\, d^2 G(z)\, H^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Consider a system of parallel wires each of diameter d and occupying a square space of side D. Let these wires carry equal currents I in the same direction. Then, if the spacing is not too close, the currents may be supposed to be concentrated on the axes and producing uniform fields acting on the other wires. The field acting upon any wire s may be written $H_s = 2Ik_s/D$ where k_s is a numerical quantity depending upon the distribution of the wires and the position of the wire s in the system. By (49) the eddy-current loss in the wire s due to the field of the neighbouring wires is

$$W = \tfrac{1}{2} R_0 k_s^{\,2} \frac{d^2}{D^2} G(z)\, I^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

per unit of length.

If there are n wires each of length l, the total loss due to the proximity of the wires will therefore be

$$\tfrac{1}{2}\mathrm{R}_0 l \frac{d^2}{\mathrm{D}^2} \mathrm{G}(z)\, \mathrm{I}^2 \sum_1^n k_s^2 ;$$

or if the wires are connected in series to give a total direct current resistance $\mathrm{R} = \mathrm{R}_0 nl$, the added resistance due to the proximity of the wires is

$$\mathrm{R}_p = u_n \mathrm{R} \frac{d^2}{\mathrm{D}^2} \mathrm{G}(z), \qquad (51)$$

in which

$$u_n \equiv \frac{1}{n} \sum_1^n k_s^2, \qquad (52)$$

and depends only on the geometry of the axes of the wires.

The total resistance (apart from capacity effects) is got by adding the skin effect to (52) so that the formula for the alternating current resistance R′ of the parallel wire system is

$$\mathrm{R}' = \mathrm{R}\left\{1 + \mathrm{F}(z) + u_n \frac{d^2}{\mathrm{D}^2} \mathrm{G}(z)\right\}. \qquad (53)$$

This formula may also be applied to circular coils if the winding section is small in comparison with the radius. It remains to determine the values of u_n.

(11) *Single Layer Systems.*—If all the axes lie in one plane (fig. 6), then, numbering the wires 1, 2, 3......s....n from left to right,

$$k_s = \frac{1}{s} + \frac{1}{s+1} + \ldots + \frac{1}{n-s}.$$

From this formula the following values of k_s^2 are calculated:—

s	$n = 4$	$n = 8$	$n = 16$	$n = 24$
1	3·35	6·70	11·00	14·00
2	0·25	2·10	5·06	7·30
3	0·25	0·61	2·82	4·63
4	3·35	0·06	1·61	3·14
5		0·06	0·89	2·16
6		0·61	0·42	1·49
7		2·10	0·14	0·98
8		6·70	0·02	0·62
9			&c.	0·36
10				0·18
11				0·06
12				0·01

These and similarly deduced results give for u_n,

n	=	2	4	6	8	10
u_n	=	1·00	1·80	2·16	2·37	2·51
n	=	12	16	24	32	inf.
u_n	=	2·61	2·74	2·91	3·00	3·29

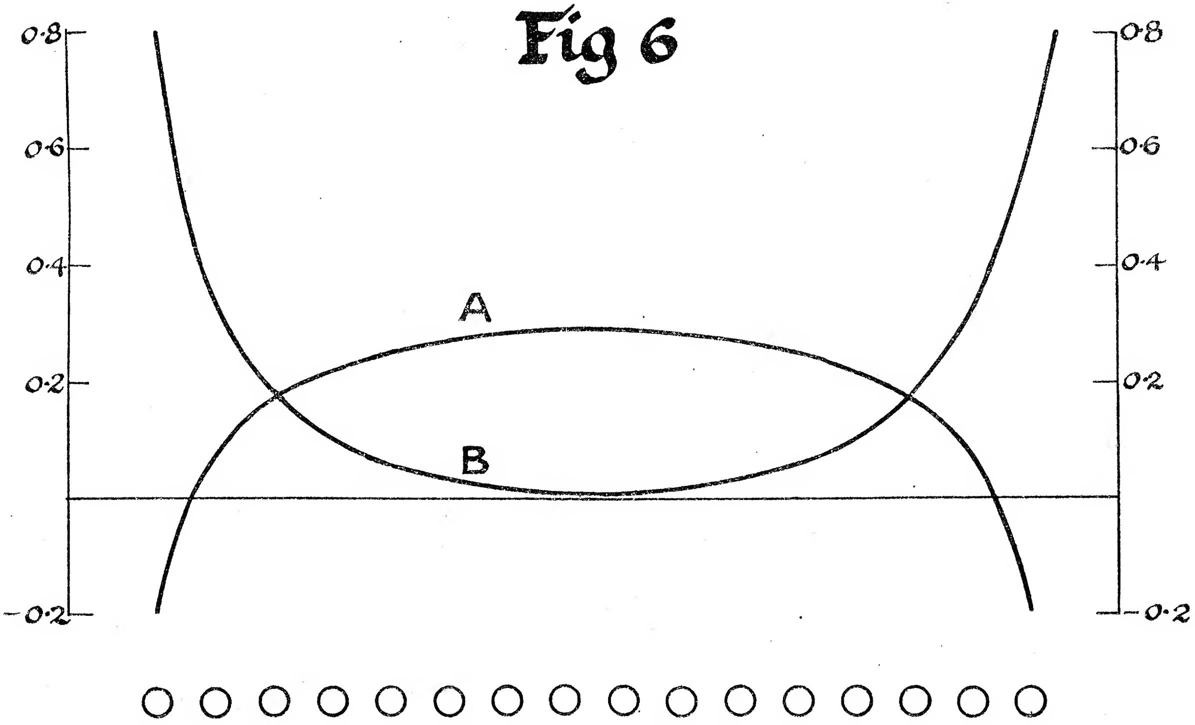


Fig. 6. Single layer 16-wire system.

Curve A shows the distribution of effective resistance produced by eddy-current losses due to the proximity of the wires.

Curve B shows the distribution of the losses throughout the system.

The value of u_n when $n = \infty$ is obtained as follows. Consider a long strip of width l composed of Nl parallel wires each carrying a current I. The field at a distance x from the edge in this strip is $2NI \log (l - x)/x$. and the mean square field is

$$H_m{}^2 = \frac{4N^2I^2}{l}\int_0^l \log^2 \frac{l-x}{x}\,dx$$

$$= 16N^2I^2\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right)$$

$$= \tfrac{4}{3}\pi^2N^2I^2.$$

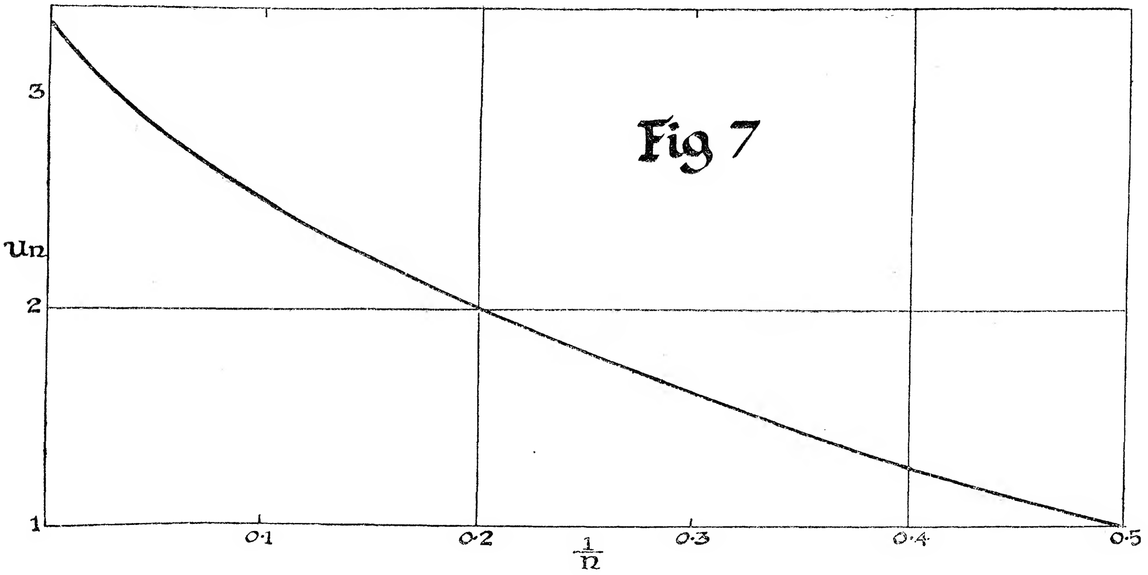
But since $N = 1/D$,

$$H_m{}^2 = 4N^2I^2u_n,$$

so that

$$Lt\, n \rightarrow \infty\ u_n = \tfrac{1}{3}\pi^2 = 3{\cdot}2899.$$

The tabulated values of u_n when plotted against $1/n$ (fig. 7) enable the value of u_n to be determined for any number of wires.



(12) *Distribution of Losses in Single Layer System.*—The energy loss in the wire s, due to the fields of the other wires, is proportional to k_s^2. From the table for k_s^2 it is seen that this is greatest in the end wires. Thus, in the case of four wires, 93 per cent. of this loss takes place in the two outer wires and only 7 per cent. in the two inner wires. For a greater number of wires, if the system is divided into four equal sections the distribution of loss is still such that approximately 93 per cent. of the loss occurs in the two outer sections. This may be of importance in measurements of effective resistance based on the determination of the increase in temperature of the system, and account should be taken of the possibility of a temperature distribution for alternating currents different from that for direct currents.

(13) *Distribution of Resistance in Single Layer System.*—The distribution of eddy-current losses throughout the system does not represent the distribution of effective resistance. In fact the energy required to produce the losses in any wire is supplied by the currents flowing in the other wires, and therefore the other wires behave as if certain resistances were added to them.

In order to determine these equivalent resistances, consider two coils (I) and (II) carrying currents I_1, I_2 which produce fields acting on a cylinder carrying no current. Ultimately these currents will be assumed equal, so that for simplicity they will be considered in phase. Let the fields in the neighbourhood of the cylinder due to these currents have intensities

$$H_1 = \alpha I_1, \quad H_2 = \beta I_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

and let them be inclined at an angle ϕ.

By (49) the eddy-current loss in the cylinder is

$$\begin{aligned} W &= \gamma (H_1^2 + H_2^2 + 2H_1H_2 \cos \phi) \\ &= \gamma (\alpha^2 I_1^2 + \beta^2 I_2^2 + 2\alpha\beta I_1 I_2 \cos \phi) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (55) \end{aligned}$$

in which

$$\gamma \equiv \tfrac{1}{8} R_0 \, d^2 G(z).$$

Now, instead of the cylinder being present, suppose the two circuits carrying the currents I_1, I_2 to be linked by the resistance system shown in fig. 8. The rate of dissipation of energy by this system is

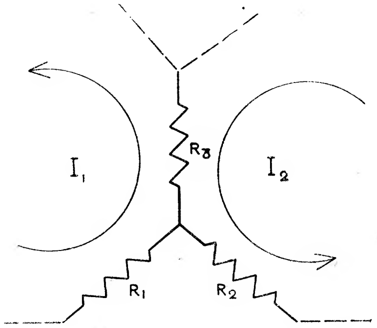


Fig 8

$$\begin{aligned} W' &= \tfrac{1}{2} \{R_1 I_1^2 + R_2 I_2^2 + R_3 (I_1 - I_2)^2\} \\ &= \tfrac{1}{2} (R_1 + R_3) I_1^2 + \tfrac{1}{2} (R_2 + R_3) I_2^2 - R_3 I_1 I_2, \quad . \; . \; . \; . \; . \quad (56) \end{aligned}$$

W′ is identical with W if the resistances have the values

$$\left. \begin{aligned} R_1 = 2\alpha (\alpha + \beta \cos \phi) \gamma, \qquad R_2 = 2\beta (\beta + \alpha \cos \phi) \gamma \\ R_3 = -2 (\alpha\beta \cos \phi) \gamma \end{aligned} \right\} \quad . \; . \; . \; . \; . \quad (57)$$

In the case $I_1 = I_2$, no current flows through R_3, so that the potential differences produced by the eddy losses in the cylinder are then such that they may be represented by the resistances R_1, R_2 in series with the respective coils.

Applying to a single layer system, suppose we require the resistance to be added to a wire s which would represent the contribution of s to the eddy losses in the whole system. Let the wire s be the coil (I), another wire r be the cylinder, and the remaining wires be the coil (II). αI is the field acting on r due to the current in s, βI is the field acting on r due to the currents in the remaining wires, so that $(\alpha + \beta \cos \phi) I$ is the nett field acting on r resolved in the direction of αI. Since in the single layer system the fields due to individual wires are collinear, $(\alpha + \beta \cos \phi) I$ is the total field in which the wire r is situated. It is to be regarded as positive when the field due to s is the same sense as the total field.

Hence if H_r is the whole field acting on r, and H_{rs} that portion of the field contributed by the wire s, the resistance to be added to the wire s to imitate the eddy losses in the remainder of the system is

$$R_{ps} = \frac{2\gamma}{I^2}\left\{\sum_{r=1}^{r=s-1} H_{rs}\cdot H_r + \sum_{r=s+1}^{r=n} H_{rs}\cdot H_r\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Remembering that

$$H_{rs} = \frac{2I}{(r-s)\,D}, \quad H_r = \frac{2I}{D}\left(\frac{1}{r}+\ldots+\frac{1}{n+r}\right),$$

(58) may be written

$$R_{ps} = u_{ns} R \frac{d^2}{D^2} G(z), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

where u_{ns} is a numerical quantity depending on the number of wires and the position of s.

The general distribution of resistance is sufficiently illustrated by the case of a 16-wire system The values of u_{ns} for this system are found by the above method to be

$s =$	1	2	3	4	5	6	7	8
	16	15	14	13	12	11	10	9
$u_{ns} =$	−0·19	+0·08	0·17	0·22	0·25	0·27	0·28	0·29

The values of u_{ns} when added should give the value of u_n for the whole system. Thus $\sum_{s=1}^{s=16} u_{ns} = 2\cdot74$, and this agrees with the tabulated value of u_n for the 16-wire system.

As regards the negative value of u_{ns} for the extreme wires, it must be remembered that the equivalent resistances for each wire are such as to imitate the *potential differences* produced by the eddy losses in the wires, and it is quite possible that the phase relations may produce a rise in potential in phase with the current in *part* of a system. The only condition that is essential is that the value of u_n for the *whole* system shall be positive. Curves A and B of fig. 6 (p. 79) show the distribution of proximity resistance and of loss respectively for a 16-wire system.

(D). SINGLE LAYER SOLID WIRE COILS.

(14) *Single Layer Coils. Effect of radius of curvature of Coils.*—A single layer circular coil, whose width of winding is small compared with the radius of the coil, differs only slightly from a straight parallel wire system, so that formula (53) will hold for a coil of this kind as a first approximation. The slight differences which occur, due to helicity of winding and owing to the fact that the wire (regarded as a cylinder under the action of a transverse field) is curved, are probably too small to be measurable and the mathematical difficulties too great to make a theoretical treatment possible.

A more important difference is the modification of the transverse field acting on the individual wires. In a straight single layer system the field acting on one wire is perpendicular to the surface of the layer, but in a single layer coil not only is this normal field modified but there also exists a component of the field acting along the surface of the layer. The case where the number of wires in the coil is large will alone be considered.

Solenoidal Coil.—Let the mutual inductance (M) between two equal parallel coaxial circles of radius a and separation b be written

$$M = 4\pi a f(2a/b). \quad \ldots \ldots \ldots \quad (60)$$

Then*

$$f(\mu) = \log 4\mu - 2 + \tfrac{3}{4}\frac{1}{\mu^2}(\log 4\mu + \tfrac{1}{3}) - \tfrac{15}{64}\frac{1}{\mu^4}(\log 4\mu - \tfrac{31}{30})$$

$$+ \tfrac{35}{256}\frac{1}{\mu^6}(\log 4\mu - \tfrac{247}{210}) + \ldots \quad \ldots \ldots \ldots \quad (61)$$

so long as $\mu < 1$.

From this expression it is readily shown by differentiation and integration that the radial and axial components of the field at the point on the prolongation of the surface of a cylindrical coil distant ξ from the edge are given by

$$H_N = 2nI\left\{f\left(\frac{2a}{\xi}\right) - f\left(\frac{2a}{b+\xi}\right)\right\}$$

$$H_T = 2nI\int_{2a/b+\xi}^{2a/\xi}\frac{d(\mu f)}{d\mu}\frac{d\mu}{\mu^2}, \quad \ldots \ldots \ldots \quad (62)$$

n being the number of turns per unit of length and I the current.

When ξ is very small, H_N tends to the value H_0—h, in which

$$H_0 = 2nI\log\frac{b+\xi}{\xi}, \quad \ldots \ldots \ldots \quad (63)$$

and is the field due to a straight strip of width b, while

$$h = 2nI\left\{\tfrac{3}{4}\frac{1}{\mu^2}(\log 4\mu - \tfrac{1}{3}) - \tfrac{15}{64}\frac{1}{\mu^4}(\log 4\mu - \tfrac{31}{30}) + \tfrac{35}{256}\frac{1}{\mu^6}(\log 4\mu - \tfrac{247}{210})\ldots\right\}. \quad (64)$$

$\mu = 2a/b$.

H_T tends to the value

$$H_T = 2nI\left\{\frac{\log 4\mu}{\mu} - \tfrac{1}{4}\frac{1}{\mu^3}(\log 4\mu - 1) + \tfrac{9}{64}\frac{1}{\mu^5}(\log 4\mu - \tfrac{7}{6}) - \tfrac{25}{256}\frac{1}{\mu^7}(\log 4\mu - \tfrac{37}{30}) + \ldots\right\} \quad (65)$$

$\mu = 2a/b$.

To find the normal and axial components at any point on the surface of the coil, divide the coil into two portions A and B to the left and right of the point in question. Then the two components of the field are

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* BUTTERWORTH, 'Phil. Mag.,' vol. 31, p. 216, 1916.

$$(H_0-H'_0)-(h-h'), \qquad H_T+H'_T$$

where the accented letters refer to the field due to the portion B, and the unaccented letters to that due to A. $H_0 - H'_0$ is the normal field for a straight strip, and $h - h'$ the correction on the normal field due to curvature, while $H_T - H'_T$ is the field tangential to the layer due to curvature.

Formulæ (64) and (65) give the following values for h and H_T :—

$b/2a$	$h/2n\mathrm{I}$	$H_T/2n\mathrm{I}$
0·1	0·025	0·369
0·2	0·079	0·595
0·3	0·149	0·767
0·4	0·229	0·902
0·5	0·313	1·010
0·6	0·400	1·096
0·7	0·486	1·167
0·8	0·569	1·225
0·9	0·651	1·273
1·0	0·728	1·313

from which the values of $h - h'$, $H_T - H'_T$ may be calculated for any point on the surface of a coil if $b/2a < 1$.

For the eddy loss formula we require the mean square field acting on the coils; that is, denoting $H_0 - H'_0$ by H, $h - h'$ by H_1, $H_T + H'_T$ by H_2, we require the mean value of

$$(H-H_1)^2+H_2^2 = H^2-2HH_1+H_1^2+H_2^2$$

throughout the surface of the coil.

As regards the integrations required in determining this mean value, the integral of H^2 leads to the straight system formula; that of H_1^2 and H_2^2 may be carried out by approximate methods, since H_1^2 and H_2^2 are finite throughout the range of integration. The integral of $H \cdot H_1$ is obtained as follows. Choose the length of the coil as twice the unit of length so that $H/2n\mathrm{I} = \log \frac{1+x}{1-x}$ at a point on the surface distant x from the centre. Suppose H_1 may be expressed in the form

$$H_1 = \alpha+\beta x+\gamma x^2+\delta x^3+\ldots,$$

The integrals required are then of the form

$$\int_{-1}^{1} x^s \log \frac{1+x}{1-x}\, dx.$$

When s is even these integrals are zero, and when s is odd have the values

$$\frac{4}{s+1}\left(1+\tfrac{1}{3}+\tfrac{1}{5}+\ldots+\frac{1}{s}\right).$$

Neglecting terms beyond x^4 in the series for H_1

$$\int_{-1}^{1} HH_1\,dx = 4nI\,(\beta+\tfrac{2}{3}\delta).$$

There is no need to evaluate β and δ for $H_1(x_1) - H_1(-x_1) = 2x_1(\beta+\delta x_1^2)$, which immediately gives $\beta+\frac{2}{3}\delta$ if we put $x_1^2 = 2/3$. Greater accuracy may be obtained by suitably choosing a series of values of x, &c., to take into account the higher terms of the series, but the above expression is sufficient for the present purpose.

The evaluation of the integrals by the above methods leads to the following table of values for u_n in applying formula (53) to solenoidal coils of length b and radius a.

SINGLE Layer Solenoidal Coils. Radius $= a$. Length $= b$. u_n in formula (53).

$b/2a$ =	0·0	0·2	0·4	0·6	0·8	1·0
u_n =	3·29	3·63	4·06	4·50	4·93	5·28

The assumption $u_n = 3{\cdot}29 + b/a$ will give results which do not differ by more than 2 per cent. from the above values.

Flat Coils.—By methods similar in principle to those used in determining the values of u_n for solenoidal coils, the following values of u_n have been found :—

SINGLE Layer Flat Coils. r = inner radius. R = outer radius.

r/R =	1·0	0·9	0·8	0·7	0·6	0·5
u_n =	3·29	3·36	3·58	3·84	4·24	4·78

(15) *Single Layer Coils at High Frequencies.*—When z is greater than 3, F(z) and G(z) assume the simplified forms

$$F(z) = (\sqrt{2}z-3)/4, \qquad G(z) = (\sqrt{2}z-1)/8,$$

so that formula (53) becomes

$$R' = \alpha+\beta z, \qquad (66)$$

in which

$$\alpha = \tfrac{1}{8}R\,(2-u_n d^2/D^2), \qquad \beta = \frac{\sqrt{2}R}{8}(2+u_n d^2/D^2).$$

Now $u_n d^2/D^2$ seldom exceeds 6, so that α/β will usually lie between $+0{\cdot}7$ and $-0{\cdot}4$.

Even when z is as small as 3, α is therefore less than $\frac{1}{4}\beta z$. For most purposes it is sufficiently accurate to take

$$R' = \beta z. \qquad (67)$$

From the relation $z^2 = 4\omega/R_0$ we may write

$$z = 0{\cdot}0479\sqrt{f/R'_0} = 830/\sqrt{R'_0\lambda}, \qquad (68)$$

in which f is the frequency in cycles per second, λ the wave-length in metres, R'_0 the resistance of the wire of the coil in ohms per 1,000 yards.

Hence (67) becomes

$$R' = A/\sqrt{\lambda} = A'\sqrt{f}, \qquad (69)$$

where A and A′ are given by

$$\left.\begin{aligned} A &= 146{\cdot}7\,(2+u_n d^2/D^2)\,R/\sqrt{R'_0} \\ A' &= 8{\cdot}47\times 10^{-3}\,(2+u_n d^2/D^2)\,R/\sqrt{R'_0} \end{aligned}\right\} \qquad (70)$$

(16) *Comparison of Formula* (69) *with Experimental Observations.*—LINDEMANN and HÜTER* have measured the effective resistances of a series of single layer coils over a range of wave-lengths to which formula (69) is applicable.

Their method was to bring the coil into resonance with an air condenser at the required wave-length and to measure the effective resistance in this condition by adding a known non-inductive resistance and observing the reduction in current. The method measures the resistance of the whole circuit of which the coil is a part, so that the resistance of the coil may be deduced if the resistances of the non-inductive portions of the circuit are known, and the condenser is assumed to be free from loss.

Their results included four solenoidal coils wound with thick solid wire, and for these coils they found that the effective resistance could be expressed in the form

$$R' = A/\sqrt{\lambda} + B/\lambda^2. \qquad (71)$$

The data given by LINDEMANN and HÜTER for these coils enable the value of A in formula (70) to be calculated. The results are given in the following Table, which includes also three other coils measured by a similar method at the National Physical Laboratory.

It is seen that the value of A as calculated from formula (70) is in good agreement with the value of A determined experimentally. In the calculation, the value of u_n taken has been the value given in the short table for solenoidal coils in Section 14. These values have been deduced upon the assumption of a large number of turns. Calculations based on the value of u_n, as deduced from a straight system containing the same number of wires as there were turns in the coil, were found to give a value of A which in every case was lower than the value deduced from observations.

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* 'Verh. Deutsch. Phys. Gesellschaft,' vol. 15, 1913, p. 219.

TABLE.—Effective Resistance of Single Layer Solenoidal Coils.

Coil.	Radius a cm.	Length b cm.	Turns n.	Diameter of wire d mm.	Direct Current R ohms.	Inductance L microhenries.	Experimental.		Calculated A.	Obsr.	Range of Wave-length metres.
							A.	B.			
1	15·5	1·8	7	2·2	—	36	9·5	$2{\cdot}14 \times 10^4$	10·4	L.H.	—
2	$9{\cdot}7_5$	2·6	11	2	0·0382	43	11·8	$2{\cdot}15 \times 10^4$	11·2	L.H.	100 3500
3	11·3	2·0	13	0·71	0·425	75	25·1	$7{\cdot}4 \times 10^4$	25·5	N.P.L.	150 600
4	10·0	6·5	18	3	—	80	14·3	$7{\cdot}0 \times 10^4$	13·2	L.H.	—
5	9·8	$2{\cdot}1_5$	15	1·1	—	84	25·5	$7{\cdot}0 \times 10^4$	25·6	L.H.	—
6	$11{\cdot}7_5$	1·0	13	0·71	0·444	100	49·0	$6{\cdot}8 \times 10^4$	50	N.P.L.	200 600
7	$8{\cdot}7_5$	7·3	40	1·62	0·193	320	56	$1{\cdot}7 \times 10^6$	55	N.P.L.	600 1500

The direct current resistances were not known in the cases of the coils 1, 4, 5. For the calculation of A, their values were deduced from the dimensions of the coils using the resistivity of copper as given by the known direct current resistance of Coil 3.

It was to throw some light upon the properties of the second term (B/λ^2) in LINDEMANN's equation that the three coils marked N.P.L. were measured. Coil No. 6, which was wound with D.S.C. wire and held together by wax and silk without any other frame, was measured first to confirm LINDEMANN's results. The observations are given below for this coil and are typical. In the Table, R_1 is the measured resistance of the coil. R' is obtained from R_1 by dividing by $(1 - \omega^2 LC)^2$, where C is the measured self capacity of the coil. The values of A and B are deduced by plotting $R'\sqrt{\lambda}$ against $\lambda^{-3/2}$ (formula (71)).

COIL No. 6. L = 100·0 microhenries. C = 20 $\mu\mu$ F.

λ metres.	R_1 ohms.	R' ohms.	$A/\sqrt{\lambda}$.	B/λ^2.	R' (formula (71)).
206	7·2	$5{\cdot}0_0$	$3{\cdot}4_2$	$1{\cdot}6_1$	$5{\cdot}0_3$
229	6·2	$4{\cdot}6_4$	$3{\cdot}2_4$	$1{\cdot}3_1$	$4{\cdot}5_5$
260	5·2	$4{\cdot}1_6$	$3{\cdot}0_4$	$1{\cdot}0_1$	$4{\cdot}0_5$
294	4·3	$3{\cdot}6_1$	$2{\cdot}8_5$	$0{\cdot}7_9$	$3{\cdot}6_4$
350	3·5	$3{\cdot}1_0$	$2{\cdot}6_4$	$0{\cdot}5_6$	$3{\cdot}2_0$
463	2·8	$2{\cdot}6_2$	$2{\cdot}2_8$	$0{\cdot}3_2$	$2{\cdot}6_0$
500	2·6	$2{\cdot}4_6$	$2{\cdot}1_9$	$0{\cdot}2_7$	$2{\cdot}4_6$
543	2·5	$2{\cdot}3_8$	$2{\cdot}1_0$	$0{\cdot}2_3$	$2{\cdot}3_3$
584	2·3	$2{\cdot}3_0$	$2{\cdot}0_2$	$0{\cdot}2_0$	$2{\cdot}2_2$

Coil No. 3 was of bare wire and supported by eight pieces of ebonite upon which equidistant grooves had been cut to keep the wires in position, these ebonite pieces being spaced equally round an octagonal wooden frame. The arrangement involved no metal except the wire of the coil. Coil 3 approximately imitates Coil 6, except that the insulation had been removed and the spacing increased. Coil No. 7 was wound on a wooden frame with D.S.C. wire and no wax. It is included in the table to increase the range of inductance and to put a severe test on the curvature correction for u_n.

In regard to the second term in formula (71), it is interesting to notice that if B is divided by L^2, the result is of the same order of magnitude for all the coils tested although the inductance increases nine fold. Thus :—

Coil No. :	1	2	3	4	5	6	7
B/L^2 =	16·3	11·6	8·5	10·9	9·9	7·2	16·7

A leakage of conductance G would contribute a term $\omega^2 L^2 G$ to the expression for the effective resistance. In terms of wave-length this becomes $3 \cdot 56\ L^2 G/\lambda^2$ if λ is in metres, L in microhenries, and G in micromhos. In order to imitate the resistance B/λ^2 by such a leakage the value of $1/G$ must range from 0·2 to 0·4 megohm to give the values observed for B.

As to whether leakage is the cause of the second term in LINDEMANN's equation, and as to whether it lies in the coil or the remainder of the circuit is a matter which requires further investigation. There is no doubt, however, that the first term of LINDEMANN's equations may be closely predicted by formula (69).

(17) *Conditions for Minimum Eddy-Current Losses in Single Layer Coils.*—The inductance of a single layer solenoidal coil of radius a and length b may be written

$$L = 4\pi a b^2 X/D^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

in which D is the distance apart of two consecutive turns and X is a function of a/b. The effective resistance of the coil is by (53)

$$R' = R\,(1 + F + u_n G d^2/D^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

where F, G depend on the frequency and diameter of wire only, while u_n is a function of a/b. The values of X and u_n for the range of a/b 1·0 to 2·4 are given below, the latter being obtained by interpolation from the table of Section 14, and the former from RAYLEIGH's formula

$$X = \log_e 8a/b - 1/2 + b^2/32a^2\,(\log_e 8a/b - \tfrac{1}{4}). \quad . \quad . \quad . \quad . \quad . \quad (74)$$

a/b =	1·0	1·2	1·4	1·6	1·8	2·0	2·2	2·4
u_n =	4·29	4·10	3·98	3·87	3·80	3·73	3·69	3·65
X =	1·651	1·816	1·958	2·084	2·195	2·296	2·388	2·466

With wire of a given diameter, the value of R′/L depends upon the values of D, a, b. If the length of wire (l) is also fixed, a may be expressed in terms of l, D and b, since

$$l = 2\pi ab/\mathrm{D}. \qquad (75)$$

Writing $\mathrm{D} = \xi d$, $b = \eta a$ we have from (72), (73), (75)

$$\mathrm{R}'/\mathrm{L} = \mathrm{R}\left(\frac{\pi d}{2l^3}\right)^{\frac{1}{2}} \frac{\xi^{\frac{1}{2}}(1+\mathrm{F}+u_n\mathrm{G}/\xi^2)}{\eta^{\frac{1}{2}}\mathrm{X}} \qquad (76)$$

The minimum value of R′/L is required for variations of ξ and η; the former gives the best spacing of the wires with a coil of given shape, and the latter gives the best shape.

Best Spacing.—(76) is minimum with regard to variation of ξ when

$$\xi^2 = 3u_n\mathrm{G}/(1+\mathrm{F}), \qquad (77)$$

and then

$$\mathrm{R}'/\mathrm{L} = \tfrac{4}{3}\mathrm{R}\left(\frac{\pi d}{2l^3}\right)^{\frac{1}{2}} \{3\mathrm{G}\,(1+\mathrm{F}\}^{\frac{1}{4}} \frac{u_n^{\frac{1}{4}}}{\eta^{\frac{1}{2}}\mathrm{X}} \qquad (78)$$

Condition (77) shows that at the best spacing the proximity losses are one-third the skin losses. If the best spacing is not employed, then, writing $\mathrm{R}'/\mathrm{L} = \tau$, and letting τ_0, ξ_0 be the values of τ, ξ when the spacing is best,

$$\tau/\tau_0 = \tfrac{1}{4}\,(\xi/\xi_0)^{\frac{1}{2}}\,\{3+(\xi_0/\xi)^2\} \qquad (79)$$

from which the following values are found:—

$\xi/\xi_0\ (\equiv \mathrm{D}/\mathrm{D}_0) =$	0·6	0·7	0·8	0·9	1·0	1·1	1·2	1·3	1·4	1·5
$\tau/\tau_0 =$	1·120	1·053	1·019	1·004	1·000	1·003	1·012	1·023	1·037	1·055

Best Shape.—Keeping the best spacing, the best shape is that value of a/b which will make $(a/b)^{\frac{1}{2}}\,u_n^{\frac{1}{4}}/\mathrm{X}$ a minimum. The following values are obtained from the table given above for u_n and X:—

$a/b =$	1·0	1·2	1·4	1·6	1·8	2·0	2·2	2·4
$\mathrm{X}/u_n^{\frac{1}{4}}\,(a/b)^{\frac{1}{2}} =$	1·147	1·165	1·172	1·174	1·172	1·167	1·161	1·152

so that if condition (77) is possible the best shape is $a/b = 1\cdot6$, and then

$$\mathrm{R}'/\mathrm{L} = 1{\cdot}872\mathrm{R}\,(d/l^3)^{\frac{1}{2}}\,\{\mathrm{G}\,(1+\mathrm{F})^3\}^{\frac{1}{4}} \qquad (80)$$

When z is very large, $\mathrm{F} = 2$, $\mathrm{G} = \sqrt{2}z/4$, so that at very high frequencies

$$\mathrm{R}'/\mathrm{L} = 0{\cdot}557\mathrm{R}z\,(d/l^3)^{\frac{1}{2}},$$

or with

$$\mathrm{R} = 4\rho l/\pi d^2, \qquad z^2 = 2\pi^2 f d^2/\rho,$$

$$\mathrm{R}'/\mathrm{L} = 3{\cdot}15\,(f\rho/ld)^{\frac{1}{2}}. \qquad (81)$$

If the frequency f is not sufficiently high for the approximation to hold, (81) must be replaced by

$$\mathrm{R}'/\mathrm{L} = \gamma(z)\,(f\rho/ld)^{\frac{1}{2}}, \quad \ldots \ldots \ldots \quad (81\mathrm{A})$$

in which $\gamma(z) \equiv 3{\cdot}15\,\{\mathrm{G}(1+\mathrm{F})^3\}^{\frac{1}{4}}/z$ and has the following values :—

$z =$	$1\cdot5$	$2\cdot0$	$2\cdot5$	$3\cdot0$	$3\cdot5$	$4\cdot0$	$4\cdot5$	$5\cdot0$	∞
$\gamma(z) =$	$3\cdot81$	$3\cdot61$	$3\cdot53$	$3\cdot48$	$3\cdot43$	$3\cdot41$	$3\cdot39$	$3\cdot37$	$3\cdot15$

It is seen that $\gamma(z)$ does not vary much throughout a large range of frequency, and if the formula

$$\mathrm{R}'/\mathrm{L} \doteq 3{\cdot}4\,\{f\rho/ld\}^{\frac{1}{2}} \quad \ldots \ldots \ldots \quad (81\mathrm{B})$$

is used this will represent the best value of alternating current time constant attainable so long as z is greater than 2. Taking for copper $\rho = 1{,}600$ C.G.S., expressing f in terms of wave-length λ (in metres), and supposing L to be in microhenries, l and d in centimetres,

$$\mathrm{R}'/\mathrm{L} = 2{\cdot}35/\sqrt{ld\lambda}.$$

Thus A in LINDEMANN'S formula cannot be less than

$$\mathrm{A}_{\mathrm{min.}} = 2\cdot35\ \mathrm{L}/\sqrt{ld}.$$

In illustration we have for coils No. 1, 2, 3 of the table of Section 16,

$$\mathrm{A}_{\mathrm{min.}} = 7\cdot1,\quad 8\cdot6,\quad 22$$

while

$$\mathrm{A}_{\mathrm{min.}}/\mathrm{A}_{\mathrm{actual}} = 0\cdot75,\quad 0\cdot73,\quad 0\cdot86.$$

The ratio $2\cdot35\ \mathrm{L}/\mathrm{A}\sqrt{ld}$ may be taken as a measure of the efficiency of any coil.

Condition that Equation (77) *may be satisfied.*—Since, if the best spacing is used, $a/b = 1\cdot6$ is always the best shape, we have from (77), with $u_n = 3\cdot87$,

$$(\mathrm{D}/d)^2 = 11{\cdot}61\mathrm{G}/1+\mathrm{F}. \quad \ldots \ldots \ldots \quad (82)$$

This gives the following values for D/d :—

$z =$	$1\cdot0$	$1\cdot5$	$2\cdot0$	$2\cdot5$	$3\cdot0$	$3\cdot5$	$4\cdot0$	$4\cdot5$	$5\cdot0$
$\mathrm{D}/d =$	$0\cdot425$	$0\cdot88_5$	$1\cdot36$	$1\cdot70_5$	$1\cdot89$	$1\cdot97$	$2\cdot01$	$2\cdot04$	$2\cdot07_5$
$z =$	$6\cdot0$	$7\cdot0$	$8\cdot0$	$9\cdot0$	$10\cdot0$	inf.			
$\mathrm{D}/d =$	$2\cdot14$	$2\cdot17$	$2\cdot20$	$2\cdot22_5$	$2\cdot24_5$	$2\cdot41$			

Now D/d must always be greater than unity in practice, so that if z is less than $1\cdot6_1$ close winding is the best. When z exceeds $1\cdot61$, spacing rapidly becomes advantageous, the best spacing at very high frequencies being $\mathrm{D} = 2\cdot4d$. As regards departure from the best spacing, the table of τ/τ_0 shows that the time constant will vary by less than 1 per cent. from the best value if D/D_0 lies between $0\cdot85$ and $1\cdot18$, by less than 5 per cent. if D/D_0 lies between $0\cdot79$ and $1\cdot28$.

If close winding is employed (keeping $a/b = 1\cdot6$), the losses at high frequencies will be **42 per cent.** greater than when the proper spacing is employed. Fig. 9 summarizes the results of this section. The full curve gives the best spacing, and the broken curves show the limits allowable if losses 1 per cent. and 5 per cent. greater than the minimum losses are permissible. The figure may be used to grade coils for different ranges of frequency.

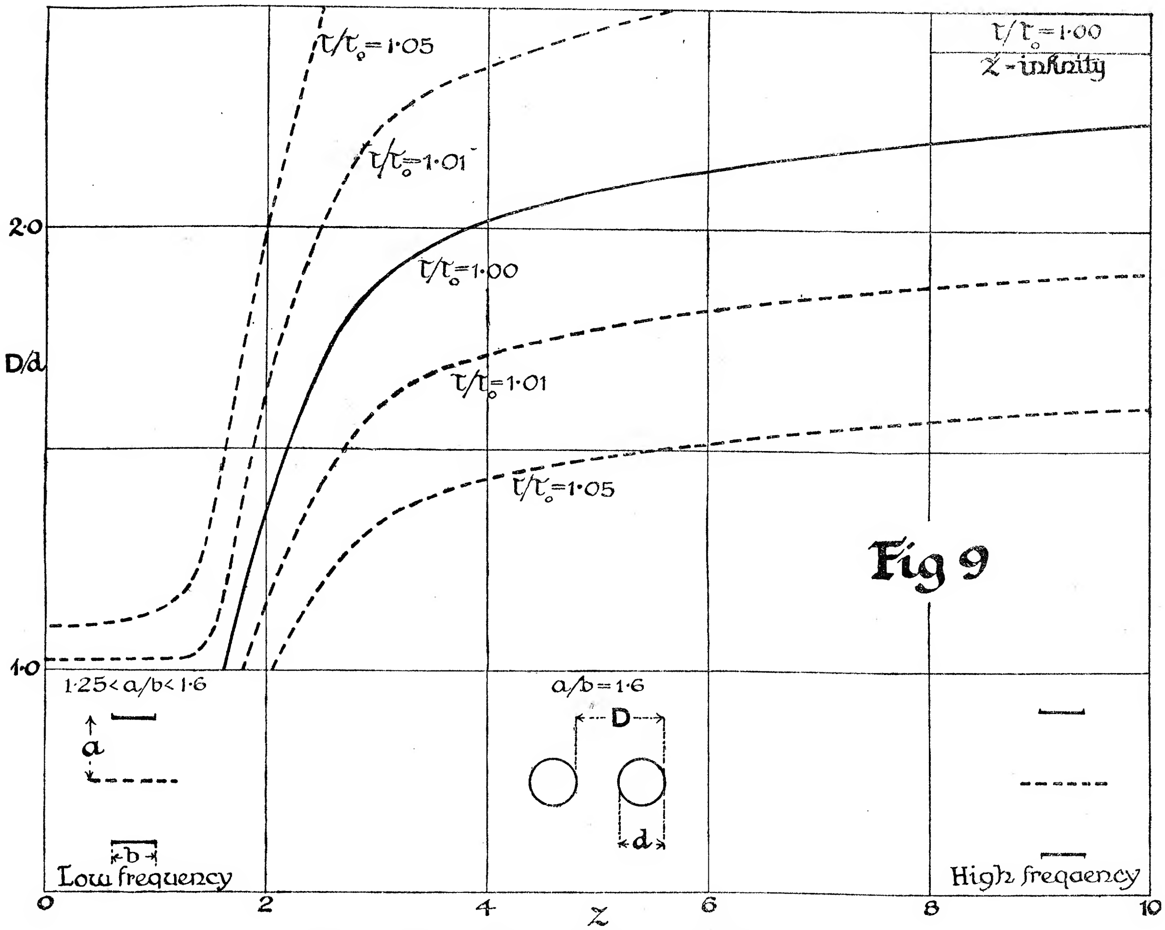

Fig. 9. Single layer solenoidal coils—Solid wire.

R' = Alternating current resistance, L = Inductance.
τ = R'/L , τ_0 = Minimum R'/L.
z = $830/\sqrt{R'_0\lambda}$, R'_0 = D.C. resistance in ohms per 1,000 yards.
λ = Wave-length in metres.
Frequency *high* if $z > 2$.
At high frequencies best shape is $a/b = 1\cdot6$, best spacing (average) is $D/d = 2$.
At low frequencies best shape (average) is $a/b = 1\cdot4$ and close winding is the best.

Thus the 5 per cent. limit will be attained if below $z = 2$ we make $a/b = 1\cdot4$* and employ close winding, and above $z = 2$ we make $a/b = 1\cdot6$ and space the wire so that

---

* Between $z = 0$ and $z = 1\cdot6$ the best value of a/b rises from $1\cdot2_5$ (the steady current value) to $1\cdot6$. The value $a/b = 1\cdot4$ will never produce a loss exceeding by $0\cdot2$ per cent. the minimum loss.

$D = 2d$. In estimating, for a given diameter of wire, the wave-length at which spacing should be employed, the formula

$$R'_0\lambda = 172{,}000 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (83)$$

(deduced from (68) with $z = 2$) is useful. Thus, with No. 28 S.W.G. wire, R'_0 in ohms per 1,000 yards is 140; so that if λ is less than 1,230 metres, spaced winding should be used.

Flat Coils.—A similar treatment for flat coils has yielded the following results:—

Spacing becomes advantageous at practically the same value of z (viz., $z = 1 \cdot 6$) as that for solenoidal coils. At any value of z greater than $1 \cdot 6$, the best ratio of inner to outer radius is $r/R = 0 \cdot 60$, and the best spacing is $1 \cdot 04$ times that for solenoidal coils. When both these conditions are satisfied the eddy-current losses in the flat coil are 5 per cent. greater than those in the solenoidal coil wound in the best way with the same wire. When z is less than $1 \cdot 6$, close winding is the best, and the best ratio of radii varies from $0 \cdot 4$ to $0 \cdot 6$. The value $r/R = 0 \cdot 5$ will therefore be the best ratio to take for a range of working from $z = 0$ to $z = 1 \cdot 6$.

(E). SINGLE LAYER STRANDED WIRE COILS.

18. *Single Layer Stranded Wire Coils.*—If we replace the solid wire in a coil by a number of insulated parallel filaments of the same copper section and connected in parallel, the direct current resistance of the coil will be unaltered, but for alternating currents the distribution of current between the filaments will not be uniform unless the filaments are interwoven in such a way that they traverse similar paths.

As a non-uniform current distribution will cause increased losses, and twisting will produce increased length in the filaments, it is a matter for investigation as to what gain may be expected by using stranded wire coils.

It will be assumed that each strand traces out a helix about the axis of stranding, the angle of which is α. Actually the radius of the helix will vary along a particular strand; but this need not be considered in getting the average result, as we may pass from one strand to its replacing strand, and thus keep at the same distance from the axis throughout the length of the stranded wire. The further assumption that α is constant throughout the section will assure that there are the same number of strands in the same axial length of any layer whatever its distance from the axis.

19. If r is the direct current resistance per unit length of one strand, the skin resistance of each strand in unit *axial* length of the wire is

$$r_0 \sec \alpha \,\{1 + F(z)\}$$

where $z^2 = \pi k \omega \delta^2$, δ being the diameter of one strand. Upon this must be superposed the resistances representing the losses due to two fields:—

(*a*) A field H_1 due to the strands in the same turn of the coil as that in which the strand in question is situated. This field is normal to the axis of stranding and tangential to the cylinder on which the strand is wound.

(*b*) A field H_2 due to the remaining turns in the coil. This field will also be assumed to be normal to the axis of stranding, and will have two components $H_2 \cos\theta$ tangential to the cylinder and $H_2 \sin\theta$ normal to the cylinder, θ being the angular position of the element under consideration.

Thus an element of one strand is situated in a nett field whose components tangential and normal to the winding cylinder are $H_1 + H_2 \cos\theta$, $H_2 \sin\theta$. These may further be resolved into components along and at right angles to the direction of the element, viz. :—

$$H_A = (H_1 + H_2 \cos\theta) \sin\alpha,$$
$$H_B = (H_1 + H_2 \cos\theta) \cos\alpha,\ H_2 \sin\theta.$$

Now, as regards the axial component of the field, it may be shown that if a cylinder is placed with its axis along the direction of an alternating field, the losses in the cylinder are one-half the losses which would occur if the cylinder were placed at right angles to the field.

Therefore the loss in an element $d\lambda$ of one strand due to the fields H_1, H_2 is by (49)

$$dW = \tfrac{1}{8} r_0 d\lambda \delta^2 G(z) (H_B^2 + \tfrac{1}{2} H_A^2).$$

Now, as we pass along one strand, the value of θ increases uniformly as we are rotating relative to the field H_2. Hence the average loss in one strand is got by replacing H_A^2, H_B^2 by their mean values throughout a complete cycle of θ; that is, H_A^2, H_B^2 are replaced by

$$\sin^2\alpha\,(H_1^2 + \tfrac{1}{2} H_2^2), \qquad \cos^2\alpha\,(H_1^2 + \tfrac{1}{2} H_2^2) + \tfrac{1}{2} H_2^2.$$

The loss per unit *axial* length of the stranded wire is thus

$$W = \tfrac{1}{8} r_0 \sec\alpha\, G(z)\, \delta^2 \{H_1^2 (1 - \tfrac{1}{2} \sin^2\alpha) + H_2^2 (1 - \tfrac{1}{4} \sin^2\alpha)\}.$$

(20) Since each strand carries the same current, the value of H_1 at a distance r from the axis is $2Ir/a_0^2$, where I is the whole current and a_0 the over-all radius of the stranded wire, the number of strands being assumed large.

Again the number of strands crossing an annular belt of width dr in the cross-section of the wire is $2rs\,dr/a_0^2$, s being the whole number of strands. The mean value of H_1^2 throughout the section is therefore

$$8I^2 \int_0^{a_0} r^3 dr / a_0^6 = 2I^2/a_0^2 = 8I^2/d_0^2,$$

in which $d_0 = 2a_0$.

The field H_2 is the same as that for the corresponding solid wire coil, and the mean value of H_2^2 throughout the coil is $4u_n I^2/D^2$, D being the distance apart of consecutive turns. Using these values in W and adding the skin losses, the following formula is obtained for the effective resistance of a stranded wire coil:—

$$R' = R_0 \sec\alpha \left[1 + F(z) + s^2\delta^2 G(z)\left\{\frac{2}{d_0^2}(1 - \tfrac{1}{2}\sin^2\alpha) + \frac{u_n}{D^2}(1 - \tfrac{1}{4}\sin^2\alpha)\right\}\right] \quad . \quad (84)$$

in which

R_0 ($\equiv r_0/s$) is the direct current resistance per unit length of the equivalent solid wire,
s = No. of strands,
δ = diameter of each strand,
α = angle of twist,
d_0 = over-all diameter of stranded wire,
D = separation of turns in coil;

while in the calculation of F and G, $z^2 = \pi k\omega\delta^2$.

(21) If the twist is so small that $\sin\alpha = \alpha$, (84) becomes

$$R' = R_0\left[1 + F + s^2\delta^2 G\left(\frac{2}{d_0^2} + \frac{u_n}{D^2}\right) + \tfrac{1}{2}\alpha^2\left\{1 + F + \tfrac{1}{2}u_n G\frac{s^2\delta^2}{D^2}\right\}\right] \quad . \quad . \quad . \quad (85)$$

The correction due to the twist will therefore be less than 1 per cent. if $\alpha < 0{\cdot}14$ radian (8°). In determining the most efficient coil only the main term in (85) need be considered.

(22) The quantities fixed will be taken to be the length of wire, the number of strands, and the diameter of each strand. Under these conditions it is clear from (85) that the best value of d_0 is $d_0 = D$, as adjustment of d_0 will have a negligible effect on the inductance. Then

$$R' = R_0\left[1 + F + \frac{s^2\delta^2}{D^2}G(2 + u_n)\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

The best value of D and shape of coil then follow by a method identical with that for the solid wire coil, except that $s\delta$ replaces d, and $2 + u_n$ replaces u_n.

The method gives as the conditions for the best time-constant

$$a/b = 1{\cdot}5 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

$$(D/s\delta)^2 = 17{\cdot}76G/(1 + F) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

and the value of the time-constant is then

$$R'/L = 1{\cdot}111\gamma(z)\sqrt{f\rho/ls\delta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

$\gamma(z)$ being calculated from the diameter of a single strand using (81A).

(23) *Limits of Application.*—The quantity $s\delta^2$ is the diameter of solid copper wire having the same section as the copper section of the stranded wire, so that if it were possible to pack the circle D entirely with copper, $D^2/s\delta^2$ could never fall below unity. Actually the limit is greater than unity, partly because the wire is circular, but also

because of twisting and the need for symmetrical distribution. Thus if three wires each of diameter δ are arranged so that their centres form an equilateral triangle, the diameter of the cylinder in which these wires can be twisted is

$$(1+2\sqrt{3})\, n\delta = 2\cdot155 n\delta,$$

in which n is a factor greater than unity, introduced to allow for insulation between contiguous wires. Denoting $D^2/s\delta^2$ by μ^2 the value of μ for the three-wire system is $1\cdot244n$. If three three-wire systems are twisted, then, assuming rigidity, the over-all diameter of the resulting nine-wire system is $(2\cdot155)^2\, n\delta$, and the value of μ is $(1\cdot244)^2\, n$.

Generally, if the operation is repeated p times, the result is a 3^p system for which

$$D = (2\cdot155)^p\, n\delta, \qquad \mu = (1\cdot244)^p\, n.$$

Applying this result to (88), it is seen that, as $G/(1 + F)$ increases with frequency, there is a lower limit of frequency below which the conditions may not be satisfied. If we depart from the condition of best internal space the resulting increase in R'/L follows a law similar to that for solid wire. If we allow a 10 per cent. variation, the actual value of μ may range between $0\cdot63\ \mu_0$ and $1\cdot75\ \mu_0$ where μ_0 is the ideal value of μ, and this may be used to extend the lower limit of the range of application. At the higher frequencies, although $G/1 + F$ tends to the finite value $1/2$, the spacing required is so large as to give unpractical coils.

If we set as practical limits to n the values $1\cdot1$ and $3\cdot3$, and allow a 10 per cent. variation, the wave-length limits for copper wire of the usual gauges used in stranding are given in the following table :—

TABLE giving Limits of Application of Formulæ (87), (88), (89).

λ = Wave-length in metres.

Wire No. S.W.G.		=	42	40	38	36
No. of strands :						
3	λ	=	0	0	0	0
	λ	=	430	630	960	1570
9	λ	=	10	10	20	30
	λ	=	600	850	1330	2150
27	λ	=	80	120	180	290
	λ	=	900	1290	2030	3200
81	λ	=	140	200	300	500
	λ	=	1270	1800	2800	4500

The shorter wave-length assumes $n = 3\cdot3$ and the longer wave-length assumes $n = 1\cdot1$. If we introduce a third system with $n = 2\cdot2$ for the mid-regions, a choice of one or other of these systems will enable the time-constant (89) to be secured to within 10 per cent. throughout the range of the table. The table shows clearly the transference of the applicability of the results to the regions of lower frequency as the stranding becomes finer. The observed inferiority of stranded wire coils at short wave-lengths is thus due to lack of internal spacing at these wave-lengths.

(24) *Comparison of Stranded Wire Coils with Solid Wire Coils.*—Assuming both coils to have the same length of wire, the same total copper section and wound to give the best *time constants*, the ratio of the time-constants is by (81A) and (89),

$$\tau'/\tau = 1\cdot111\gamma(z)/s^{\frac{1}{4}}\gamma(s^{\frac{1}{2}}z), \quad \ldots \ldots \ldots \quad (90)$$

since $s\delta^2 = d^2$ and z is proportional to d.

In (90) $\tau' = \mathrm{R}'/\mathrm{L}$ for the stranded wire coil and τ that for the solid wire. Now, the ratio $\gamma(z)/\gamma(s^{\frac{1}{2}}z)$ lies between 1 and $2^{\frac{1}{4}}$ for all possible values of z and s, so that the formula

$$\tau'/\tau \doteqdot 1\cdot2/s^{\frac{1}{4}} \quad \ldots \ldots \ldots \ldots \quad (91)$$

may be taken as comparing the two cases.

For the 3-system we have therefore

s	=	3	9	27	81
τ'/τ	=	0·91	0·69	0·53	0·40

when the same length of wire is used in both coils.

If coils of equal *inductance* are compared, the conditions are different, as the spacing for stranded wire coils is not the same as that for solid wire coils. In fact, throughout the range for which spacing is advantageous,

$$\mathrm{D} \doteqdot 2d$$

for the solid wire coils, and for stranded wire coils on the 3-system having the same copper section,

$$\mathrm{D} = (1\cdot244)^p\, nd.$$

For coils of the same shape and of radius a, the inductance L is proportional to a^3/D^2 and the length of wire l is proportional to a^2/D. Hence, to keep the inductance constant, l^3/D must be constant. We have then

$$l \propto \mathrm{D}^{1/3}, \quad a \propto \mathrm{D}^{2/3},$$

and from (89)

$$\mathrm{R}'/\mathrm{L} \propto \mathrm{D}^{-1/6}.$$

Thus, if accented letters refer to the stranded wire coils, the relative values for equal inductance are

$$(l'/l)^3 = (a'/a)^{3/2} = (1 \cdot 244)^p \, n/2$$

and

$$\tau'/\tau \doteqdot 1 \cdot 2/s^{1/4} \{(1 \cdot 244)^p \, n/2\}^{1/6},$$

in the extreme case, when $s = 81$, $p = 4$, $n = 3 \cdot 3$, $l' = 1 \cdot 58l$, $a' = 2 \cdot 5a$, $\tau'/\tau = 0 \cdot 32$.

(25) *Modification of Formulæ when Strands are few in number.*—The value of the field H_1 was calculated on the assumption of a large number of strands, giving $H_1^2 = 8I^2/d_0^2$. If there are two touching strands each of diameter δ, the field acting on one strand due to a current $I/2$ in the other is I/δ, and d_0 (the diameter of the circumscribing circle) is 2δ. Hence, for this case, $H_1^2 = 4I^2/d_0^2$. The factor $2/d_0^2$ in the formula (84) must therefore be replaced by $1/d_0^2$. With three strands whose centres form an equilateral triangle of side δ, $H_1^2 = 4I^2/3\delta^2$ and $d_0 = 2 \cdot 155\delta$, so that $2/d_0^2$ is replaced by $1 \cdot 55/d_0^2$.

With four strands in square order, the centres form a square of side δ, $H_1^2 = 9I^2/8\delta^2$ and $d_0 = 2 \cdot 414\delta$; $2/d_0^2$ is replaced by $1 \cdot 65/d_0^2$. These new values react on the conditions (87), (88), (89); the "shape" condition (87) is practically unaltered. The "spacing" condition (88) gives slightly too high a value for D, viz.:—

$S = 2$	3	4
$D/D_0 = 0 \cdot 90$	$0 \cdot 96$	$0 \cdot 97$

where D_0 is the value calculated by (88) and D the true value.

The factor $1 \cdot 111$ in (89) must be replaced by $1 \cdot 054$, $1 \cdot 078$, $1 \cdot 084$ when $s = 2, 3, 4$. These differences are small, so that the theory may be safely applied even when the strands are few.

(F). MANY-LAYERED COILS.

(26) *Coils of Many Layers.*—Let the winding section of the coil be $b \times c$. Let c/b be small, and let b be small compared with the radius of the coil. Let there be m layers in the depth c and n turns per layer.

The field at any point in the section will have two components H_b and H_c parallel to b and c respectively, which will act independently in producing eddy-current losses.

As regards H_c, the field acting on a single wire is the same as that for a single layer coil for which $D = b/mn$. Thus the added resistance due to the action of H_c is

$$\tfrac{1}{3}\pi^2 R G(z) \, (mnd/b)^2,$$

d being the diameter of the wire and R the direct current resistance of the coil.

As regards H_b, each layer behaves as a current sheet having current density nI/b.

In the immediate neighbourhood of the sheet the component of the field parallel to the sheet is

$$h = 2\pi n\mathrm{I}/b$$

and reverses its direction as we pass through the sheet.

Assuming h to be constant throughout the depth c, the field acting on the first layer is $(m - 1)\,h$, on the second layer $(m - 3)\,h$, and generally on the r^{th} layer $(m - 2r - 1)\,h$. The mean square value for all the layers is therefore

$$h^2\,\{(m-1)^2+(m-3)^2+(m-s)^2+\ldots\}/m = (m^2-1)\,h^2/3 = \tfrac{4}{3}\pi^2\,(m^2-1)\,(n\mathrm{I}/b)^2.$$

Upon applying this result to (49) it follows that the added resistance due to the action of H_b is

$$\tfrac{1}{3}\pi^2\,(m^2-1)\,\mathrm{RG}\,(z)\,(nd/b)^2.$$

Adding these resistances to the skin resistance, the formula for the resistance of a many-layered coil is

$$\mathrm{R}' = \mathrm{R}\,\{1+\mathrm{F}+\tfrac{1}{3}\pi^2\,(2m^2-1)\,(nd/b)^2\,\mathrm{G}\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (92)$$

The corresponding formula for a stranded wire coil is obtained by replacing F by $\mathrm{F} + 2s^2\delta^2\mathrm{G}/d_0^{\,2}$ and d by $s\delta$. F and G in this case are calculated, using the diameter of a single strand.

Assuming that the correction for curvature for the many-layered coil is of the same form as that for the single layer coil, the following formula includes all the previous formulæ—

$$\mathrm{R}' = \mathrm{R}\ (1 + \mathrm{F} + \mathrm{MG}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (93)$$

in which for solid wire coils

$$\mathrm{M} = u_n\,(2m^2-1)\,(nd/b)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (94)$$

and for stranded wire coils

$$\mathrm{M} = 2\,(s\,\delta/d_0)^2+u_n\,(2m^2-1)\,(ns\,\delta/b^2)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (95)$$

(27) *Best Conditions for Many-layered Coils.*—If different coils are wound with the same length and diameter of wire on the same shape of frame, and with the same spacing between the wires but with different radii, then the inductance will vary as $m^{\frac{1}{2}}$ while the resistance will be of the form

$$\alpha + \beta m^2,$$

in which

$$\alpha = \mathrm{R}\,\{1+\mathrm{F}-u_n\,(nd/b)^2\,\mathrm{G}\}, \qquad \beta = 2u_n\,(nd/b)^2\,\mathrm{G}.$$

At low frequencies F and G are negligibly small, so that increasing the number of layers will always improve the time-constant. At high frequencies the best time-constant is obtained when

$$\alpha = 3\beta m^2,$$

which gives

$$6m^2+1 = (1+\mathrm{F})/\mathrm{G}u_n(nd/b)^2. \qquad (96)$$

Assuming the condition such as to make m large,

$$6\,(mnd/b)^2 \doteqdot (1+\mathrm{F})/u_n\mathrm{G}, \qquad (97)$$

an expression determining the total number of turns ($\mathrm{N} = m \times n$). With this value of N we obtain from

$$\mathrm{L} = 4\pi\mathrm{N}^2a\mathrm{X}\left(\frac{a}{b}\right), \qquad l = 2\pi a\mathrm{N},$$

the relation

$$\mathrm{L}^2 = \frac{2l^3}{\pi d}\,\frac{b}{a}\,\mathrm{X}^2\left(\frac{1+\mathrm{F}}{6u_n\mathrm{G}}\right)^{\frac{1}{2}}$$

for the inductance of the coil, and this leads to the same value for a/b as for single-layer coils, viz.:—

$$a/b = 1\cdot6, \qquad u_n = 3\cdot87. \qquad (98)$$

When both conditions are satisfied

$$\mathrm{R}'/\mathrm{L} = 1\cdot187\gamma(z)\sqrt{f\rho/ld}. \qquad (99)$$

When $z > 1$, condition (97) with $u_n = 3\cdot87$ shows that $(mnd/b)^2 < 3$; and since nd/b is of the order unity, m will not exceed 2. Many-layered coils are therefore only of advantage when $z < 1$. When this is the case, $\mathrm{G} = z^4/64$ and F is negligible.

Condition (97) may then be written, when $a/b = 1\cdot6$,

$$\mathrm{N}d/b = 1\cdot66/z^2;$$

or, expressing z in terms of wave-length and diameter, and assuming the wire to be of copper of resistivity 1600 C.G.S. units,

$$\mathrm{N} = 2\cdot8\times10^{-4}\lambda a/d^3, \qquad (100)$$

λ being the wave-length in metres, a the coil radius in centimetres, and d the diameter of the wire in millimetres.

For stranded wire coils of the same total copper section the conditions are

$$a/b = 1\cdot6$$

$$\mathrm{N} = 2\cdot8\times10^{-4}\lambda a\sqrt{s}/d^3, \qquad (101)$$

while

$$\mathrm{R}'/\mathrm{L} = 1\cdot187\gamma(z)\sqrt{f\rho/ls\delta}, \qquad (102)$$

d being the diameter of the equivalent solid copper, δ the diameter of one strand, s the number of strands, and z is calculated from the diameter of a single strand.

Thus the gain in time-constant by using stranded wire is $1/s^{\frac{1}{4}}$; but, in addition, a larger number of turns, and therefore an increased inductance, may be obtained with stranded wire, while maintaining the best conditions.

(28) *Design of Coils of Large Inductance.*—If a coil of large inductance is required to have minimum effective resistance at a specified wave-length, the conditions

$$a/b = 1 \cdot 6,$$

$$\mathrm{N} = 2 \cdot 8 \times 10^{-4} \lambda a \sqrt{s}/d^3, \qquad \ldots \ldots \ldots \ldots \qquad \text{(A)}$$

together with the formula for the inductance

$$\mathrm{L} = 25 \cdot 5 \mathrm{N}^2 a, \qquad \ldots \ldots \ldots \ldots \qquad \text{(B)}$$

determine the radius, shape and number of turns for a given diameter of wire.

Usually these coils are required to resonate with a condenser of given capacity. In this case, if C is the resonating capacity in micro-microfarads,

$$\lambda^2 = 3 \cdot 55 \times 10^{-3} \mathrm{LC}. \qquad \ldots \ldots \ldots \ldots \qquad \text{(C)}$$

Eliminating L and λ^2 between (A), (B), (C), we find

$$a^3 = 1 \cdot 4 \times 10^8 d^6 s \mathrm{C},$$

a relation independent of the number of turns. Thus, whatever inductance is used, the coils must all have the same radius if wound with the same type of wire. In illustration, let the wire consist of nine strands, each of diameter 0·2 mm., and let the resonating capacity be 1,000 $\mu\mu$ F.

Then

$$s = 9, \qquad d = \sqrt{s}\delta = 0 \cdot 6,$$

from which

$$a \doteqdot 9 \text{ cm.}$$

Thus, if $\mathrm{L} = 20 mh$, $\mathrm{N} = 297$. As the winding length $b = 5 \cdot 6$ cm., this could be arranged by having 6 layers of 50 turns each. To avoid large self-capacities the winding should be "sliced."